THEOREM

LET $F: B^n \to B$ BE A BOOLEAN FUNCTION. THEN ITS DUAL F^d IS GIVEN BY THE FORMULA

$$F^d(x_1,\ldots,x_n) = \overline{F(\bar{x}_1,\ldots,\bar{x}_n)} \quad .$$

RMK: THE DUALITY PRINCIPLE FOLLOWS IMMEDIATELY FROM THIS RESULT SINCE THEN $F = G$ IFF $F^d = G^d$.

WE LEAVE THE PROOF OF THE ABOVE THEOREM AS AN EXERCISE (PROBLEM #27 P. 708). HINT: USE THE RECURSIVE DEFINITION OF BOOLEAN EXPRESSION AND INDUCTION ON THE EXPRESSION WHICH REPRESENTS F. THE KEY STEP IS TO USE THE IDENTITIES $\overline{\bar{x}\bar{y}} = x+y$ AND $\overline{\bar{x}+\bar{y}} = xy$.

EX $F(x,y,z) = x\bar{y} + z\bar{x}$

$$\overline{F(\bar{x},\bar{y},\bar{z})} = \overline{\bar{x}\bar{\bar{y}} + \bar{z}\bar{\bar{x}}} = \left(\overline{\bar{x}y}\right)\cdot\left(\overline{\bar{z}x}\right)$$

$$= (\bar{\bar{x}}+\bar{y})\cdot(\bar{\bar{z}}+\bar{x}) = (x+\bar{y})(z+\bar{x})$$

$$= F^d(x,y,z) \quad .$$

10.2 REPRESENTING BOOLEAN FUNCTIONS

DEFN

A LITERAL IS A BOOLEAN VARIABLE OR ITS COMPLEMENT: x OR $\bar{x}$. A MINTERM OF THE VARIABLES $x_1, x_2, \ldots, x_n$ IS A PRODUCT OF THE FORM

$$y_1 y_2 \cdots y_n$$

WHERE EACH y_i IS EITHER x_i OR $\bar{x}_i$. i.e. A MINTERM IS A PRODUCT OF LITERALS

NOTE THAT A MINTERM IS TRUE FOR EXACTLY ONE COMBINATION OF VALUES OF ITS VARIABLES

EX. $x\bar{y}\bar{z}$ IS A MINTERM OF x, y, z

x	y	z	$\bar{y}$	$\bar{z}$	$x\bar{y}\bar{z}$
0	0	0	1	1	0
0	0	1	1	0	0
0	1	0	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	0	1	1	0	0
1	1	0	0	1	0
1	1	1	0	0	0

OBSERVE THAT EVERY BOOLEAN FUNCTION CAN BE REPRESENTED AS A SUM OF MINTERMS OF ITS VARIABLES. THIS EXPRESSION IS CALLED THE <u>SUM OF PRODUCTS</u> REPRESENTATION OR <u>DISJUNCTIVE NORMAL FORM (DNF)</u> OF THE FUNCTION.

EX.
$$\begin{aligned} F(x,y,z) &= \bar{x}(y+\bar{z}) \\ &= \bar{x}y + \bar{x}\bar{z} \\ &= \bar{x}\cdot y\cdot 1 + \bar{x}\cdot 1\cdot \bar{z} \\ &= \bar{x}y(z+\bar{z}) + \bar{x}(y+\bar{y})\bar{z} \\ &= \bar{x}yz + \bar{x}y\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} \\ &= \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} \end{aligned}$$

TO FIND THE DNF OF A FUNCTION, GIVEN ITS TRUTH TABLE, WRITE THE MINTERM CORRESPONDING TO EACH 1 IN THE OUTPUT COLUMN, THEN TAKE THE SUM OF THESE MINTERMS.

EX.

x	y	z	G(x,y,z)	
0	0	0	0	
0	0	1	1	$\longrightarrow \bar{x}\bar{y}z$
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	1	$\longrightarrow x\bar{y}z$
1	1	0	1	$\longrightarrow xy\bar{z}$
1	1	1	0	

$$\therefore\ G(x,y,z) = \bar{x}\bar{y}z + x\bar{y}z + xy\bar{z}$$

THIS PROCEDURE SHOWS THAT ANY BOOLEAN FUNCTION CAN BE PLACED IN DNF, AND IN PARTICULAR THE MAPPING

$$\{\text{EXPRESSIONS}\} \longrightarrow \{\text{FUNCTIONS}\}$$

IS ONTO. NOTE ALSO THAT THE DNF OF A FUNCTION IS UNIQUE UP TO ORDER OF TERMS.

LIKEWISE ANY BOOLEAN FUNCTION CAN BE REPRESENTED BY A PRODUCT OF SUMS EXPRESSION CALLED THE CONJUCTIVE NORMAL FORM (CNF).

THIS FORM CAN BE OBTAINED BY TAKING DUALS TWICE AND USING $F^{dd} = F$.

EX. $F(x,y) = xy + \bar{x}\bar{y}$

$$\begin{aligned}(xy + \bar{x}\bar{y})^d &= (x+y)(\bar{x}+\bar{y}) \\ &= x\bar{x} + \bar{x}y + x\bar{y} + y\bar{y} \\ &= 0 + \bar{x}y + x\bar{y} + 0 \\ &= \bar{x}y + x\bar{y}\end{aligned}$$

$$\begin{aligned}(xy + \bar{x}\bar{y})^{dd} &= (\bar{x}y + x\bar{y})^d \\ &= (\bar{x}+y)(x+\bar{y})\end{aligned}$$

$$\therefore F(x,y) = (\bar{x}+y)(x+\bar{y}).$$

SINCE EVERY BOOLEAN FUNCTION CAN BE REPRESENTED USING THE OPERATORS $\cdot, +, \bar{\ }$ WE SAY THAT THE SET

$$\{\cdot, +, \bar{\ }\}$$

IS <u>FUNCTIONALLY COMPLETE</u>.

ACTUALLY THIS IS NOT THE SMALLEST SUCH SET. USING THE IDENTITY $x+y = \overline{\bar{x}\,\bar{y}}$ WE CAN ELIMINATE ALL OCCURANCES OF $+$, SHOWING THAT $\{\cdot, \bar{\ }\}$ IS ALSO FUNCTIONALLY COMPLETE.

LIKEWISE $xy = \overline{\bar{x} + \bar{y}}$ SHOWS THAT $\{+, \bar{\ }\}$ IS FUNCTIONALLY COMPLETE AS WELL.

ITS ALSO TRUE THAT THERE ARE COMPLETE SETS CONSISTING OF JUST ONE OPERATOR.

<u>DEFN</u>

x	y	NAND $x \mid y$	NOR $x \downarrow y$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

OBSERVE THAT $\{\mid\}$ IS FUNCTIONALLY COMPLETE SINCE

$$\bar{x} = x \mid x \quad \& \quad xy = (x \mid y) \mid (x \mid y)$$

AND THE FACT THAT $\{\cdot, ^{-}\}$ IS FUNCTIONALLY COMPLETE.

LIKEWISE $\{\downarrow\}$ IS COMPLETE SINCE

$$\bar{x} = x \downarrow x \quad \& \quad x + y = (x \downarrow y) \downarrow (x \downarrow y)$$

AND THE FACT THAT $\{+, ^{-}\}$ IS COMPLETE.

EXERCISE: PROVE ALL OF THE ABOVE IDENTITIES.

EXERCISE:
FIND THE DNF OF $x \mid y$ AND $x \downarrow y$, AND SHOW THAT

$$(x \downarrow y)^{d} = x \mid y$$