PROOF:

LET m BE THE NUMBER OF SUCH PERMUTATIONS

MARK ALL OBJECTS SO AS TO DISTINGUISH BETWEEN THEM. THERE ARE $n!$ PERMUTATIONS OF THESE MARKED OBJECTS.

THESE PERMUTATIONS CAN BE CONSTRUCTED AS FOLLOWS. FOR EACH OF THE m (INDISTINGUISHABLE) PERMUTATIONS, ARRANGE THE OBJECTS OF EACH TYPE IN ALL POSSIBLE WAYS. BY THE PRODUCT RULE WE CAN PERFORM THIS CONSTRUCTION IN $m \cdot n_1! \cdot n_2! \cdots n_k!$ WAYS. THUS

$$m \cdot n_1! \cdot n_2! \cdots n_k! = n!$$

$$\therefore \quad m = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

///

EX. FIND THE NUMBER OF BIT STRINGS OF LENGTH 10 WHICH CONTAIN EXACTLY 6 ONES.

$$\left.\begin{array}{l} n=10 \\ n_1=6 \\ n_2=4 \end{array}\right\} \quad \frac{10!}{6!\,4!} = 210$$

RECALL WE SOLVED THIS SAME PROBLEM EARLIER USING BINOMIAL COEFFICIENTS:

$$\binom{10}{6} = \frac{10!}{6!\,4!} = 210$$

EX. FIND THE NUMBER OF TERNARY STRINGS OF LENGTH 20 WHICH CONTAIN EXACTLY 3 ZEROS, 12 ONES, AND 5 TWO'S.

$$\frac{20!}{3!\,12!\,5!} = 7{,}054{,}320$$

MULTINOMIAL COEFFICIENTS

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!} \qquad (n_1 + \cdots + n_k = n).$$

MULTINOMIAL THEOREM

LET $x_1, x_2, \ldots, x_k \in \mathbb{R}$. THEN

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

WHERE THE SUM IS OVER ALL K-TUPLES $(n_1, n_2, \ldots, n_k)$ SUCH THAT $n_1 + n_2 + \cdots + n_k = n$, AND $n_i \geq 0$ FOR $1 \leq i \leq k$.

5.1 Discrete Probability

Definitions

An experiment is a procedure which yields one of a set of possible outcomes. Such a set is called a sample space. An event is a subset of the sample space.

Let S be a finite sample space, and $E \subseteq S$ an event. The probability of E is

$$P(E) = \frac{|E|}{|S|} .$$

Note that this is not the only possible definition of probability. It is based on the assumption that all outcomes are in some sense symmetrical, i.e. equally likely.

Observe

- $\forall E \subseteq S : \; 0 \leq P(E) \leq 1$
- $P(\emptyset) = 0$
- $P(S) = 1$

<u>EX.</u> A SINGLE DIE IS THROWN. WHAT IS THE PROBABILITY THAT THE NUMBER SHOWING IS AT LEAST 5 ?

$$S = \{1,2,3,4,5,6\}, \quad E = \{5,6\}$$

$$\therefore P(E) = \frac{2}{6} = \frac{1}{3} = .333\ldots$$

<u>EX</u> TWO DICE ARE THROWN. WHAT IS THE PROBABILITY THAT THE SUM OF THE NUMBERS SHOWING IS 8 ?

$$S = \left\{ \begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{array} \right\} \quad E$$

$$P(E) = \frac{5}{36} = .1388\ldots$$

EX WHAT IS THE PROBABILITY THAT A 5 CARD POKER HAND CONTAINS 4 OF A KIND?

SUITS

KINDS

	H	D	C	S
A				
K				
Q				
J				
10				
9				
8				
7				
6				
5				
4				
3				
2				

H = HEARTS, D = DIAMONDS } RED

C = CLUBS, S = SPADES } BLACK

$S = \{$ALL 5 CARD POKER HANDS$\}$

$E = \{$HANDS CONTAINING 4 OF A KIND$\}$

$$|S| = \binom{52}{5} = 2{,}598{,}960$$

$$|E| = \binom{13}{1} \cdot \binom{4}{4} \cdot \binom{48}{1} = 624$$

$$P(E) = \frac{1}{4165} = .00024\ldots$$

EX. WHAT IS THE PROBABILITY THAT A 5 CARD POKER HAND CONTAINS EXACTLY 3 OF ONE KIND (i.e. NOT 4 OF A KIND).

$$|E| = \binom{13}{1} \cdot \binom{4}{3} \cdot \binom{48}{2} = 58656$$

$$P(E) = \frac{94}{4165} = .02256\ldots$$

COMBINATIONS OF EVENTS

THEOREM

GIVEN $E \subseteq S$, THE PROBABILITY OF THE EVENT $\bar{E} = S - E$ IS

$$P(\bar{E}) = 1 - P(E)$$

PROOF:

$$P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - P(E). \quad ///$$

EX WHAT IS THE PROBABILITY THAT A 5 CARD POKER HAND CONTAINS AT LEAST ONE ACE?

LET $E = \{\text{NO ACES}\}$ SO THAT $\bar{E} = \{\text{AT LEAST ONE ACE}\}$

THEN

$$P(\bar{E}) = 1 - P(E) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

$$= 1 - \frac{35673}{54145} = .34115\ldots$$

THEOREM

GIVEN EVENTS $E_1 \subseteq S$, $E_2 \subseteq S$, THEN

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

PROOF:

DIVIDE $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$ BY $|S|$.

///

EX. WHAT IS THE PROBABILITY THAT A 5 CARD POKER HAND CONTAINS A RED ACE?

LET $E_1 = \{AH\}$, $E_2 = \{AD\}$. THEN

$|E_1| = |E_2| = \binom{51}{4}$ AND $|E_1 \cap E_2| = \binom{50}{3}$.

$$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{245}{1326} = .18476\ldots$$

Ex. What is the probability that a 5 card poker hand contains either 3 of a kind or 4 of a kind.

Let $E_1 = \{\text{exactly 3 of a kind}\}$ and $E_2 = \{\text{exactly 4 of a kind}\}$. Then $E_1 \cap E_2 = \phi$ and

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= (.02256..) + (.00024..) - 0$$

$$= .02280...$$

Ex To win a super lottery you must pick 7 correct numbers out of 40. What is the probability of winning.

$$|S| = \binom{40}{7}, \quad |E| = 1$$

$$P(E) = \frac{1}{18,643,560} = (5.363..) \cdot 10^{-8}$$

In general

$$P(\text{win}) = \frac{\#\text{ ways of winning}}{\#\text{ ways of playing}}$$

EX. IN ANOTHER STATE, THE LOTTERY COMMISSION PICKS 12 NUMBERS FROM THE SET $\{1, \ldots, 60\}$. TO WIN YOU MUST PICK 6 OF THOSE 12.

$$P(\text{win}) = \frac{\binom{12}{6}}{\binom{60}{6}} = \frac{30}{1{,}625{,}450} = (1.845..)\cdot 10^{-5}$$