EX. $P(x) =$ '$x^2 < 0$' (UNIVERSE $\mathbb{R}$)

$\exists x\, P(x) =$ 'THERE IS AT LEAST ONE REAL NUMBER WHOSE SQUARE IS NEGATIVE'

WHAT IS THE TRUTH VALUE OF THIS PROPOSITION? WHAT HAPPENS IF WE REPLACE THE UNIVERSE $\mathbb{R}$ BY $\mathbb{C}$?

CONSIDER THE MEANING OF QUANTIFICATION IN A FINITE UNIVERSE, SAY $U = \{a, b\}$.

$$\forall x\, P(x) \equiv P(a) \wedge P(b)$$
$$\exists x\, P(x) \equiv P(a) \vee P(b).$$

THEN

$$\begin{aligned} \neg \forall x\, P(x) &\equiv \neg \left(P(a) \wedge P(b)\right) \\ &\equiv \neg P(a) \vee \neg P(b) \\ &\equiv \exists x\, \neg P(x). \end{aligned}$$

SIMILARLY $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$.

THESE IDENTITIES ALSO HOLD FOR INFINITE UNIVERSES.

STATEMENT	TRUE WHEN	FALSE WHEN
$\forall x\, P(x)$	$P(x)$ TRUE FOR ALL x	$P(x)$ FALSE FOR AT LEAST ONE x
$\exists x\, P(x)$	$P(x)$ TRUE FOR AT LEAST ONE x	$P(x)$ FALSE FOR ALL x

A VARIABLE x IN A PROPOSITIONAL FUNCTION IS SAID TO BE <u>BOUND</u> IF IT HAS EITHER BEEN QUANTIFIED, OR HAS BEEN ASSIGNED A VALUE. OTHERWISE x IS CALLED <u>FREE</u>

<u>Ex.</u> $P(x,y) =$ '$x+y = y+x$'
$Q(y,z) =$ '$y-z = z-y$'

- $\forall x \forall y\, P(x,y)$ (x, y BOUND)
- $\forall x\, P(x,y)$ (x BOUND, y FREE)
- $P(x,y)$ (x, y FREE)

- $\forall x \exists y\, P(x,y) \vee Q(y,z)$ (z FREE)
- $\exists z \exists x\, P(x,3) \wedge Q(3,z)$ (x, y, z BOUND)
- $\forall z\, Q(y,z)$ (z BOUND, y FREE)

ANY EXPRESSION WHICH CONTAINS FREE VARIABLES IS NOT A PROPOSITION SINCE IT'S TRUTH VALUE DEPENDS ON THE VALUES OF THOSE VARIABLES.

CONSIDER THE SENTENCE

'ALL HORSES CALIFORNIA ARE BLUE'

HOW CAN WE EXPRESS THIS SYMBOLICALLY USING QUANTIFIERS ?

__EX__ LET $U = \{\text{HORSES IN CALIFORNIA}\}$ AND $B(x) =$ 'x IS BLUE'. THEN

$$\forall x \; B(x)$$

__EX.__ LET $U = \{\text{HORSES}\}$ AND $C(x)$ = 'x LIVES IN CALIFORNIA'. THEN

$$\forall x \left(C(x) \rightarrow B(x) \right)$$

__EX.__ LET $U = \{\text{LIVING THINGS}\}$ AND $H(x)$ = 'x IS A HORSE'. THEN

$$\forall x \left(H(x) \wedge C(x) \rightarrow B(x) \right).$$

__EX.__ (WRONG) $\forall x \left(H(x) \wedge C(x) \wedge B(x) \right)$.
WHAT DOES THIS SAY ?

__READ__: EXAMPLES FROM LEWIS CARROLL P.37.

(1.4) NESTED QUANTIFIERS

LET $P(x,y)$ BE ANY PROPOSITIONAL FUNCTION ON ANY UNIVERSE. THEN

$$\forall x \forall y\, P(x,y) \equiv \forall y \forall x\, P(x,y)$$
$$\exists x \exists y\, P(x,y) \equiv \exists y \exists x\, P(x,y)$$

BUT $\forall x \exists y\, P(x,y) \not\equiv \exists y \forall x\, P(x,y)$

TO ILLUSTRATE THE LAST STATEMENT LET $P(x,y) = 'x < y'$ (UNIVERSE $\mathbb{R}$). THEN

$\forall x \exists y\, P(x,y)$ IS TRUE
$\exists y \forall x\, P(x,y)$ IS FALSE

<u>EX.</u> $L(x,y) = $ 'x LOVES y', $U = \{\text{PEOPLE}\}$.

'EVERYBODY LOVES SOMEBODY' $\equiv \forall x \exists y\, L(x,y)$

'SOMEBODY LOVES EVERYBODY' $= \exists x \forall y\, L(x,y)$

'NOBODY LOVES EVERYBODY' $=$

$$\neg \exists x \forall y\, L(x,y) \equiv \forall x \neg \forall y\, L(x,y)$$
$$\equiv \forall x \exists y\, \neg L(x,y).$$

'THERE is EXACTLY ONE PERSON WHOM EVERYBODY LOVES'.

$$\exists y \Big(\forall x\, L(x,y) \wedge \forall z \big(z \neq y \rightarrow \exists w\, \neg L(w,z) \big) \Big).$$

(1.5) Methods of Proof

Our formal rules of inference will have the following form

$$\begin{array}{l} \text{Hypothesis 1} \\ \underline{(\text{Hypothesis 2})} \\ \therefore \text{Conclusion} \end{array}$$

Each rule is based on a corresponding tautology of the form

$$(H_1 \wedge H_2) \to C$$

Running example

p = 'it is raining today'
q = 'class is cancelled'
r = 'we go to the beach'

Addition

$$\begin{array}{l} \underline{p} \\ \therefore p \vee q \end{array} \qquad p \to (p \vee q)$$

Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array} \qquad (p \wedge q) \to p$$

CONJUNCTION

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & p \wedge q \end{array} \qquad (p \wedge q) \rightarrow (p \wedge q)$$

MODUS PONENS

$$\begin{array}{ll} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array} \qquad [p \wedge (p \rightarrow q)] \rightarrow q$$

MODUS TOLLENS

$$\begin{array}{ll} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array} \qquad [\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$$

HYPOTHETICAL SYLLOGISM

$$\begin{array}{ll} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array} \qquad [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

DISJUNCTIVE SYLLOGISM

$$\begin{array}{ll} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array} \qquad [(p \vee q) \wedge \neg p] \rightarrow q$$

RESOLUTION

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array} \qquad [(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

A PROOF CONSISTS OF A SEQUENCE OF PROPOSITIONS BEGINING WITH ONE OR MORE HYPOTHESES WHICH ARE ASSUMED TO BE TRUE AND ENDING WITH A CONCLUSION. EACH PROPOSITION IN THE SEQUENCE WHICH IS NOT A HYPOTHESIS MUST FOLLOW FROM PREVIOUS PROPOSITIONS USING VALID INFERENCE RULES, LOGICAL EQUIVALENCES, OR PREVIOUSLY PROVEN STATEMENTS.

EX.

$$\left.\begin{array}{l} 1.)\ \neg p \vee \neg q \rightarrow r \wedge s \\ 2.)\ s \rightarrow u \\ 3.)\ \neg u \end{array}\right\}$$ HYPOTHESES CONCLUSION: p

4.) $\neg s$ MODUS TOLLENS 2, 3

5.) $\neg r \vee \neg s$ ADDITION 4, COMMUTATIVE

6.) $\neg(r \wedge s)$ DeMORGAN 5

7.) $\neg(\neg p \vee \neg q)$ MODUS TOLLENS 1, 6

8.) $p \vee$ DeMORGAN, DOUBLE NEGATION 7

9.) p SIMPLIFICATION 8

Ex.

1.) $\neg p \wedge q$
2.) $r \to p$
3.) $\neg r \to s$
4.) $s \to t$

Hypotheses (1.–4.) Conclusion: t

5.) $\neg p$ Simplification 1
6.) $\neg r$ Modus Tollens 2,5
7.) s Modus Ponens 3,6
8.) t Modus Ponens 4,7

Some Fallacies

Ex. Affirming the Conclusion
'If it rains today, then class is cancelled.'
'Class is cancelled'
$\therefore$ 'It is raining today'

$[(p \to q) \wedge q] \to p$ is NOT a tautology.

Ex. Denying the Hypothesis
'If it rains today then class is cancelled'
'It is not raining today'
$\therefore$ 'Class is not cancelled'

$[(p \to q) \wedge \neg p] \to \neg q$ is NOT a tautology

INFERENCE RULES FOR QUANTIFIERS

UNIVERSAL INSTANTIATION

$$\begin{array}{l} \underline{\forall x\, P(x)} \\ \therefore\ P(c) \end{array}$$

ALL HORSES ARE BLUE
∴ TRIGGER IS BLUE

UNIVERSAL GENERALIZATION

$$\begin{array}{l} \underline{P(c) \text{ FOR AN ARBITRARY } c} \\ \therefore\ \forall x\, P(x) \end{array}$$

AN ARBITRARILY SELECTED HORSE IS BLUE
∴ ALL HORSES ARE BLUE

EXISTENTIAL INSTANTIATION

$$\begin{array}{l} \underline{\exists x\, P(x)} \\ \therefore\ P(c) \text{ FOR SOME } c \end{array}$$

THERE EXISTS A BLUE HORSE
FOR SOME HORSE C, C IS BLUE

EXISTENTIAL GENERALIZATION

$$\begin{array}{l} \underline{P(c) \text{ FOR SOME } c} \\ \therefore\ \exists x\, P(x) \end{array}$$

TRIGGER IS BLUE
∴ THERE EXISTS A BLUE HORSE