4.2 THE PIGEONHOLE PRINCIPLE

EX.

A DRAWER CONTAINS A DOZEN BROWN SOCKS AND A DOZEN BLACK SOCKS (UNMATCHED.) A MAN TAKES OUT SOCKS AT RANDOM IN THE DARK. HOW MANY SOCKS MUST BE REMOVED TO BE SURE HE HAS (AT LEAST) 2 OF THE SAME COLOR?

ANSWER: 3

WHAT IS THE SMALLEST NUMBER OF SOCKS WHOSE REMOVAL WOULD GUARANTEE THAT THE MAN HAS 6 OF THE SAME COLOR?

ANSWER: 11

THEOREM: PIGEONHOLE PRINCIPLE (PP)
IF $K+1$ (OR MORE) OBJECTS ARE PLACED IN K BOXES, THEN (AT LEAST) ONE BOX CONTAINS (AT LEAST) 2 OBJECTS.

PROOF

ASSUME, TO GET A CONTRADICTION, THAT ALL K BOXES CONTAIN FEWER THAN 2 OBJECTS.
$\therefore$ ALL BOXES CONTAIN AT MOST 1 OBJECT
$\therefore$ THE NUMBER OF OBJECTS IS AT MOST THE NUMBER OF BOXES K. THIS CONTRADICTS THAT THERE ARE $K+1$ OBJECTS. ///

Ex.

IN ANY GROUP OF 13 PEOPLE, AT LEAST 2 WERE BORN IN THE SAME MONTH.

Ex.

HOW MANY STUDENTS MUST BE ENROLLED IN A UNIVERSITY TO GUARANTEE THAT AT LEAST 100 WERE BORN IN THE SAME MONTH?

ANSWER: 1189

THEOREM: GENERALIZED PIGEONHOLE PRINCIPLE (GPP)

IF n (OR MORE) OBJECTS ARE PLACED IN k BOXES, THEN (AT LEAST) ONE BOX CONTAINS (AT LEAST) $\lceil \frac{n}{k} \rceil$ OBJECTS.

IN THE PREVIOUS EXAMPLE, WE SEEK THE SMALLEST n FOR WHICH

$$\left\lceil \frac{n}{12} \right\rceil = 100$$

i.e. $$99 < \frac{n}{12} \leq 100$$

i.e. $$12 \cdot 99 < n \leq 12 \cdot 100 .$$

THE SMALLEST SUCH n IS OBVIOUSLY $n = 12 \cdot 99 + 1 = 1189$.

PROOF: (OF GPP)

ASSUME, TO GET A CONTRADICTION, THAT EACH BOX CONTAINS AT MOST $(\lceil \frac{n}{k} \rceil - 1)$ OBJECTS. THEN THE NUMBER OF OBJECTS n MUST SATISFY:

$$n \leq k \cdot \left(\lceil \tfrac{n}{k} \rceil - 1 \right) < k\left(\left(\tfrac{n}{k} + 1 \right) - 1 \right) = k \cdot \frac{n}{k} = n,$$

WHICH SAYS $n < n$, A CONTRADICTION.
$\therefore$ SOME BOX MUST CONTAIN AT LEAST $\lceil \frac{n}{k} \rceil$ OBJECTS. ///

NOTE: WE USED $\lceil x \rceil < x + 1 \quad \forall x \in \mathbb{R}$.

EX.

THERE ARE 38 DIFFERENT TIME PERIODS DURING WHICH CLASSES AT A UNIVERSITY MAY BE SCHEDULED. IF THERE ARE 677 DIFFERENT CLASSES, HOW MANY ROOMS ARE NEEDED?

BY THE GPP AT LEAST ONE PERIOD MUST HAVE $\lceil 677/38 \rceil = \lceil 17.8 \rceil = 18$ CLASSES ASSIGNED TO IT. THEREFORE AT LEAST 18 ROOMS ARE NECESSARY.

Ex.

FIND THE SMALLEST n SUCH THAT IN ANY GROUP OF n PEOPLE, AT LEAST 100 HAVE THE SAME BIRTHDAY.

WE SEEK THE SMALLEST n SUCH THAT
$\left\lceil \frac{n}{366} \right\rceil = 100$

i.e. $99 < \frac{n}{366} = 100$

$366 \cdot 99 < n \leq 366 \cdot 100$

$\therefore \quad n = 366 \cdot 99 + 1 = 36{,}235$.

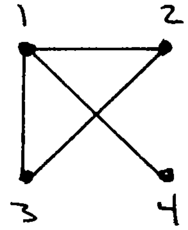
THEOREM

IN ANY GROUP OF 6 PEOPLE IN WHICH EACH PAIR ARE EITHER FRIENDS OR ENEMIES, THERE MUST EXIST EITHER A GROUP OF 3 MUTUAL FRIENDS OR A GROUP OF 3 MUTUAL ENEMIES.

IT IS HELPFUL TO STATE THIS THEOREM IN TERMS OF GRAPH THEORY.

A <u>GRAPH</u> G CONSISTS OF A PAIR (V, E) OF SETS CALLED <u>VERTICES</u> AND <u>EDGES</u> RESPECTIVELY. WE REQUIRE $V \neq \phi$. EACH EDGE $e \in E$ IS AN UNORDERED PAIR OF VERTICES: $e = \{x, y\} \subseteq V$. WE WRITE $e = xy$.

EX.



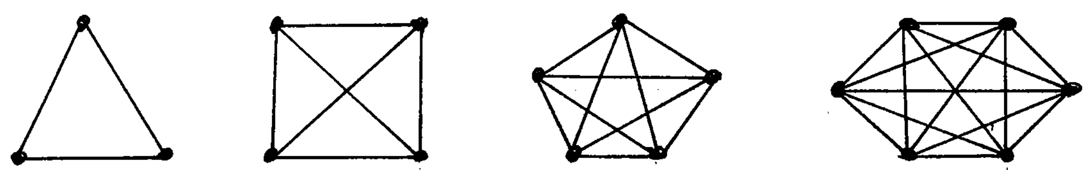
$V = \{1, 2, 3, 4\}$

$E = \{12, 13, 14, 23\}$.

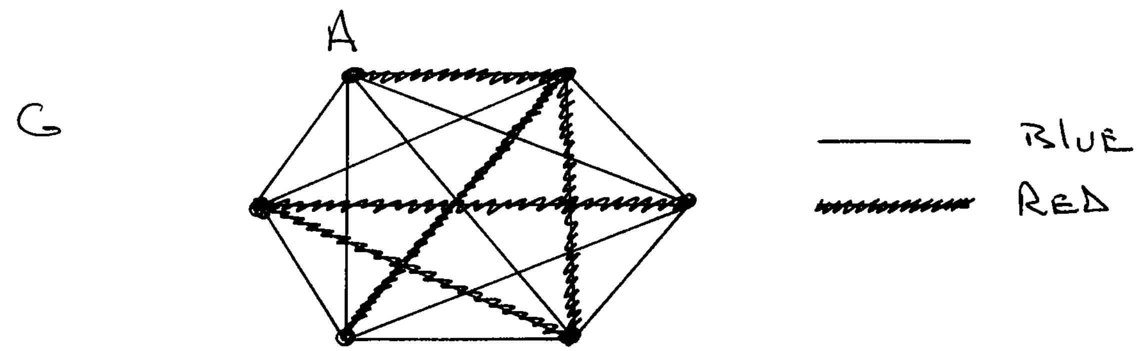
WE SAY THAT VERTICES 1 AND 2 ARE ADJACENT, THAT EDGE 12 JOINS 1 TO 2, AND THAT EDGE 12 IS INCIDENT WITH VERTEX 1.

THE COMPLETE GRAPH ON n VERTICES HAS AN EDGE JOINING EACH PAIR OF VERTICES

$n=3$ $\quad$ $n=4$ $\quad$ $n=5$ $\quad$ $n=6$



THE PRECEDING THEOREM CAN NOW BE STATED AS FOLLOWS: LET G BE THE COMPLETE GRAPH ON 6 VERTICES, AND SUPPOSE EACH EDGE OF G IS COLORED EITHER BLUE (FRIENDSHIP) OR RED (ENMITY). THEN G MUST CONTAIN EITHER A BLUE TRIANGLE OR A RED TRIANGLE.



PROOF OF THM.
LET A BE A PERSON IN THE GROUP. THE REMAINING 5 PEOPLE FALL INTO ONE OF 2 CLASSES: $F = \{\text{FRIENDS OF } A\}$, $E = \{\text{ENEMIES OF } A\}$.

BY THE P.P. AT LEAST $\lceil \frac{5}{2} \rceil = 3$ PEOPLE ARE IN THE SAME CLASS.

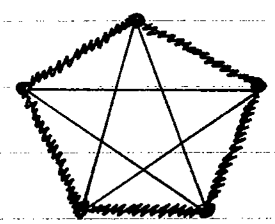
CASE 1 : $|F| \geq 3$
SAY B, C, D ARE FRIENDS OF A. IF ANY TWO OF THESE, SAY B, C, ARE FRIENDS (OF EACH OTHER) THEN A, B, C FORMS A GROUP OF 3 MUTUAL FRIENDS. OTHERWISE B, C, D FORM A GROUP OF 3 MUTUAL ENEMIES.

CASE 2 : $|E| \geq 3$
FOLLOWS FROM CASE 1 BY SWAPPING 'FRIEND' $\Leftrightarrow$ 'ENEMY'

IN BOTH CASES THERE IS EITHER A GROUP OF 3 MUTUAL FRIENDS OR 3 MUTUAL ENEMIES. ///

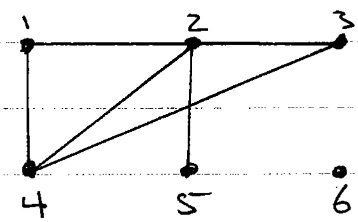
NOTE: 6 is the smallest number of people for which this conclusion is true. For example, it is possible that in a group of 5 people (in which each pair are either friends or enemies), there exists neither 3 mutual friends nor 3 mutual enemies.

EX



DEFN: Let $G = (V, E)$ and $x \in V$. The degree of x is the number of edges incident with x.

EX



$\deg(1) = 2$ $\deg(2) = 4$
$\deg(4) = 3$ $\deg(6) = 0$

THEOREM
Let G be a graph on $n \geq 2$ vertices. Then G contains 2 vertices of like degree.

PROOF: Exercise
Consider subgraph $(V - V_0, E)$ where $V_0 = \{\text{vertices of degree } 0\}$, then apply the Pigeonhole Principle.