7.6 PARTIAL ORDERS

<u>DEFN:</u>

A RELATION $R \subseteq S \times S$ IS CALLED A <u>PARTIAL ORDER</u> IF IT IS REFLEXIVE, ANTI-SYMMETRIC AND TRANSITIVE. THE SET S TOGETHER WITH R IS CALLED A <u>PARTIALLY ORDERED SET</u> OR <u>POSET</u>, AND DENOTED (S, R)

<u>EX.</u> $(\mathbb{Z}, \leq)$ AND $(\mathbb{Z}, \geq)$ ARE POSETS.

<u>EX.</u> $(\mathbb{Z}^+, |)$ IS A POSET

CHECK: $\forall a, b, c \in \mathbb{Z}^+$

- $a | a$
- $a | b \wedge b | a \rightarrow a = b$
- $a | b \wedge b | c \rightarrow a | c$

~~EX. LET A BE A SET AND S = P(A). THEN (S, ⊆) IS A POSET.~~

<u>EX.</u> LET S BE ANY SET. THEN $(\mathbb{P}(S), \subseteq)$ IS A POSET.

CHECK: $\forall A, B, C \in \mathbb{P}(S)$.

- $A \subseteq A$
- $A \subseteq B \wedge B \subseteq A \rightarrow A = B$
- $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

<u>EX.</u> LET $S = \{1, 2, 3, 6\}$ AND

$$R = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$$

THEN (S, R) IS A POSET. NOTE THAT R IS JUST THE DIVISIBILITY RELATION RESTRICTED TO S.

<u>NOTATION</u>

WE WILL USE $\preceq$ TO STAND FOR A GENERIC PARTIAL ORDER INSTEAD OF R. NOTE THE SIMILARITY WITH $\leq$ AND $\subseteq$.

WE WRITE $(S, \preceq)$ FOR A GENERAL POSET. ALSO FOR $a, b \in S$ WRITE

$$a \prec b \iff a \preceq b \text{ AND } a \neq b.$$

NOTE: $\prec$ IS TRANSITIVE AND ANTISYMMETRIC.

DEFN

LET $(S, \preceq)$ BE A POSET, AND $a, b \in S$. WE SAY a AND b ARE COMPARABLE IF EITHER $a \preceq b$ OR $b \preceq a$. OTHERWISE a AND b ARE CALLED INCOMPARABLE

EX IN $(\mathbb{Z}^+, |)$: 5 AND 10 ARE COMPARABLE WHILE 7 AND 15 ARE NOT.

EX. IN $(\mathbb{P}(\{1,2,3\}), \subseteq)$: $\{1\}$, $\{1,2\}$ ARE COMPARABLE WHILE $\{1,2\}$ AND $\{1,3\}$ ARE NOT.

DEFN.

LET $(S, \preceq)$ BE A POSET. IF ANY TWO ELEMENTS OF S ARE COMPARABLE WE SAY S IS A TOTALLY ORDERED SET OR LINEARLY ORDERED SET. IN THIS CASE $\preceq$ IS CALLED A TOTAL ORDER OR LINEAR ORDER. $(S, \preceq)$ IS ALSO SOMETIMES CALLED A CHAIN.

EX. BOTH $(\mathbb{Z}, \leq)$ AND $(\mathbb{Z}, \geq)$ ARE TOTALLY ORDERED.

<u>Ex.</u> $(\mathbb{Z}^+, |)$ is NOT TOTALLY ORDERED SINCE IT CONTAINS INCOMPARABLE ELEMENTS.

<u>Ex.</u> $(\mathbb{P}(S), \subseteq)$ is NOT TOTALLY ORDERED UNLESS $|S| \leq 1$.

<u>DEFN</u>

LET $(S, \preceq)$ BE A POSET. WE SAY $(S, \preceq)$ is <u>WELL ORDERED</u> IF IT is TOTALLY ORDERED AND EVERY NON-EMPTY SUBSET OF S HAS A "LEAST" ELEMENT.

i.e. IF $A \subseteq S$ AND $A \neq \phi$ THEN:

$$\exists a \in A, \forall b \in A : a \preceq b.$$

<u>Ex.</u> $(\mathbb{Z}^+, \leq)$ is WELL ORDERED.

$(\mathbb{Z}, \leq)$ is NOT WELL ORDERED.

Also $(\mathbb{Z}^-, \leq)$ is NOT WELL ORDERED

BUT $(\mathbb{Z}^-, \geq)$ <u>is</u> WELL ORDERED.

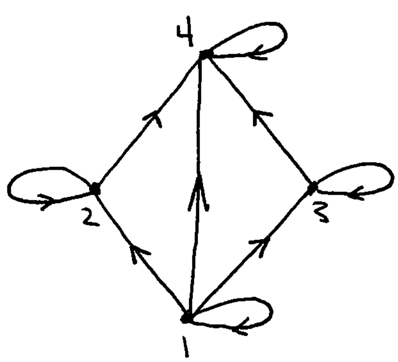
READ P. 417: LEXOGRAPHIC ORDER

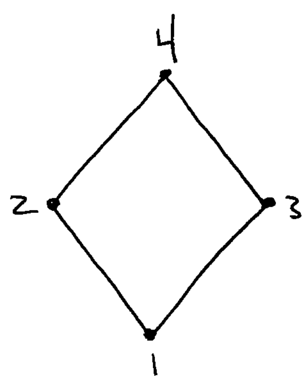
HASSE DIAGRAMS

THE HASSE DIAGRAM OF A POSET IS OBTAINED FROM ITS DIGRAPH REPRESENTATION BY

- ERASING ALL LOOPS
- ERASING DIRECTED EDGES WHICH ARISE FROM TRANSITIVITY
- ERASE DIRECTIONS

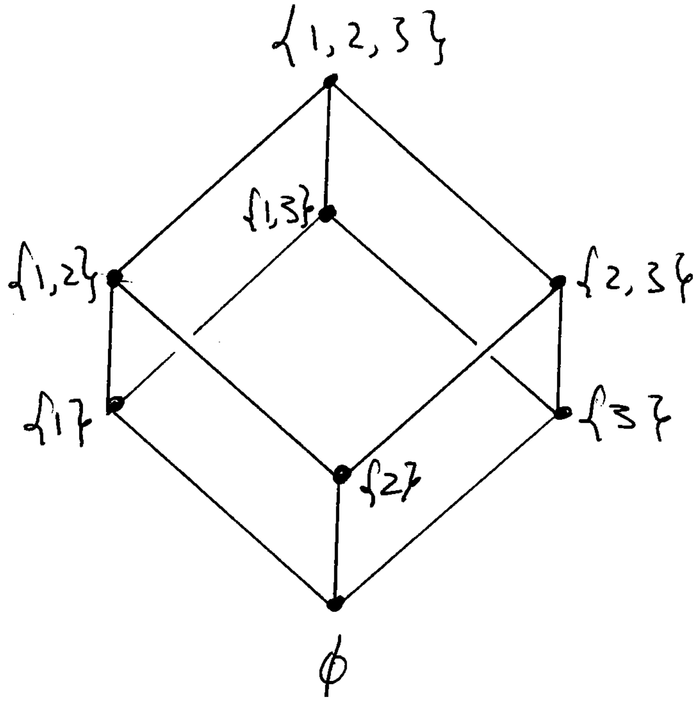
EX $(\{1,2,3,6\}, |)$



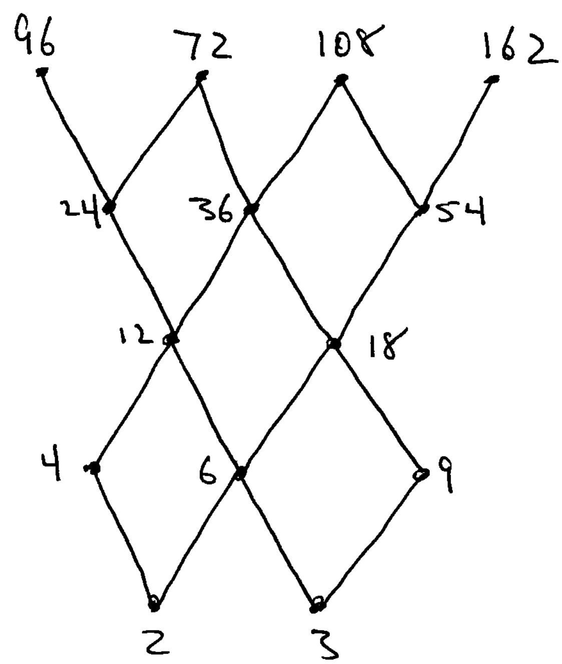


CONVENTION: $x \preceq y$ IFF THERE IS AN UPWARD PATH FROM x TO y IN THE HASSE DIAGRAM.

Ex. $(\mathcal{P}(\{1,2,3\}), \subseteq)$



Ex. $(\{2,3,4,6,9,12,18,24,36,54,72,96,108,162\}, \mid)$



Defn:
Let $(S, \preceq)$ be a poset. An element $a \in S$ is called maximal if there is no $b \in S$ such that $a \prec b$. i.e.

$$\neg \exists b \in S : a \prec b$$

In other words for all $b \in S$, either $b \preceq a$ or a and b are incomparable

a is called a greatest element (or maximum) if for all $b \in S$: $b \preceq a$. Note that a greatest element is unique if it exists, while there may be many maximal elements.

$a \in S$ is called minimal if there is no $b \in S$ such that $b \prec a$.

i.e. $\neg \exists b \in S : b \prec a$
i.e. $\forall b \in S$: $a \preceq b$ or a & b are incomparable

$a \in S$ is called a least element (or minimum) if $\forall b \in S$: $a \preceq b$.

Note a maximum (resp. minimum) element is necessarily maximal (resp. minimal), but not converseley.

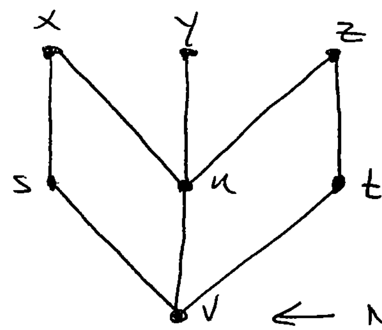
Ex. $(\mathbb{P}(\{1,2,3\}), \subseteq)$

$\{1,2,3\}$ is maximum
ϕ is minimum
There are no other maximal or minimal elements.

Ex. $(\{2, 3, \ldots, 162\}, \mid)$

2, 3 are minimal
96, 72, 108, 162 are minimal
There are no maximum or minimum elements.

Ex.



← maximal

← minimal (& least).

DEFN.

LET $(S, \preceq)$ BE A POSET AND $A \subseteq S$.
$u \in S$ IS CALLED AN UPPER BOUND OF A
IF

$$\forall a \in A : \quad a \preceq u .$$

SIMILARLY $l \in S$ IS A LOWER BOUND OF A
IF

$$\forall a \in A : \quad l \preceq a$$

$x \in S$ IS CALLED THE LEAST UPPER BOUND OF A
IF x IS AN UPPER BOUND, AND IS ~~A~~ THE
LEAST ELEMENT AMONG ALL UPPER BOUNDS
OF A:

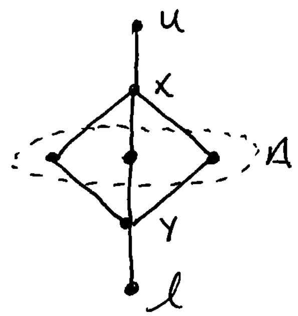
$$\forall a \in A : a \preceq x \text{ AND } \forall \text{ U.B. } z \text{ OF } A : \; x \preceq z$$

SIMILARLY $y \in S$ IS THE GREATEST LOWER BOUND OF A

WRITE $lub(A)$ AND $glb(A)$ FOR THE
LEAST UPPER BOUND AND GREATEST LOWER BOUND.
IF THEY EXIST.

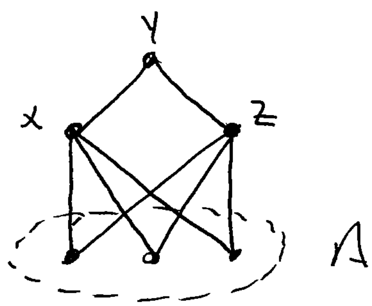
NOTE THAT IF $lub(A)$ AND $glb(A)$ EXIST
THEY ARE UNIQUE (PROB. 34.a)

<u>Ex.</u>



x, u ARE UPPER BOUNDS OF A. x = lub(A)
y, l ARE LOWER BOUNDS OF A. y = glb(A).

<u>Ex.</u>



x, y, z ARE UPPER BOUNDS OF A, BUT
A HAS NO LEAST UPPER BOUND.