IN FACT THE CONVERSE IS ALSO TRUE!

GIVEN A PARTITION $\mathcal{P} = \{S_1, \ldots, S_n\}$ OF S, DEFINE $R \subseteq S \times S$ BY

$$R = \{(x,y) \in S \times S : \exists i \text{ s.t. } x \in S_i \wedge y \in S_i\}.$$

i.e. xRy IFF x AND y BELONG TO THE SAME MEMBER OF $\mathcal{P}$.

NOTE EACH $x \in S$ BELONGS TO AT LEAST ONE S_i BY (3), AND TO AT MOST ONE BY (2). THUS EACH $x \in S$ BELONGS TO EXACTLY ONE MEMBER OF $\mathcal{P}$.

<u>THEOREM.</u>
GIVEN A PARTITION $\mathcal{P}$ ON S, THE RELATION R ABOVE IS AN EQUIVALENCE RELATION ON S WHOSE EQUIVALENCE CLASSES ARE PRECISELY THE MEMBERS OF $\mathcal{P}$.

<u>PROOF:</u>
EVERY $x \in S$ IS IN THE SAME MEMBER OF $\mathcal{P}$ AS ITSELF, SO xRx, AND R IS REFLEXIVE.

ALSO x IS IN SAME MEMBER OF $\mathcal{P}$ AS y IFF y IS IN SAME MEMBER OF $\mathcal{P}$ AS x. THUS $xRy \Leftrightarrow yRx$, WHENCE R IS SYMMETRIC.

FINALLY IF x AND y ARE IN THE SAME MEMBER OF $\mathcal{P}$, AND y AND z ARE IN THE SAME MEMBER OF $\mathcal{P}$, THEN x AND z ARE IN THE SAME MEMBER OF $\mathcal{P}$. THUS $xRy \wedge yRz \Rightarrow xRz$, AND R IS TRANSITIVE.

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HOW MANY EQUIVALENCE RELATIONS ARE THERE ON A FINITE SET?

DEFINE

$$B_n = (\#\text{ EQ. RELATIONS ON AN } n\text{-ELEMENT SET})$$

BY CONVENTION $B_0 = 1$. (THE EMPTY PARTITION IS THE ONLY PARTITION OF $\emptyset$.)

THE SEQUENCE $\{B_n\}_{n=0}^{\infty}$ IS CALLED THE <u>BELL NUMBERS</u>.

THEOREM

For all $n \geq 1$:

$$B_n = \sum_{k=1}^{n} C(n-1, k-1) B_{n-k}$$

For example:

$B_0 = 1$

$B_1 = C(0,0) B_0 = 1 \cdot 1 = 1$

$B_2 = C(1,0) B_1 + C(1,1) B_0 = 1 \cdot 1 + 1 \cdot 1 = 2$

$B_3 = C(2,0) B_2 + C(2,1) B_1 + C(2,2) B_0$

$= 1 \cdot 2 + 2 \cdot 1 + 1 = 5$

$B_4 = C(3,0) \cdot B_3 + C(3,1) B_2 + C(3,2) B_1 + C(3,3) B_0$

$= 1 \cdot 5 + 3 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 = 15$

Ex. $S = \{1, 2, 3\}$

$P_1 = \{\{1\}, \{2\}, \{3\}\}$

$P_2 = \{\{1, 2\}, \{3\}\}$

$P_3 = \{\{1, 3\}, \{2\}\}$

$P_4 = \{\{2, 3\}, \{1\}\}$

$P_5 = \{\{1, 2, 3\}\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$R_3 = \{(1,1), (1,3), (3,1), (3,3), (2,2)\}$

$R_4 = \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

EXERCISE

LIST ALL 15 PARTITIONS (AND EQUIVALENCE RELATIONS) OF $\{1, 2, 3, 4\}$.

PROOF OF THM

LET $S = \{1, 2, \dots, n\}$ AND LET P BE A PARTITION OF S. EXACTLY ONE MEMBER OF P CONTAINS n, CALL IT X:

$$X \subseteq S \quad \text{AND} \quad n \in X.$$

LET $Y = X - \{n\}$ SO THAT $X = Y \cup \{n\}$ AND $Y \subseteq \{1, 2, \dots, n-1\}$. THE REMAINING MEMBERS OF P FORM A PARTITION OF THE SET

$$\{1, 2, \dots, n-1\} - Y \quad .$$

THESE DATA

- THE SUBSET $Y \subseteq \{1, \dots, n-1\}$
- THE PARTITION OF $\{1, \dots, n-1\} - Y$

UNIQUELY DETERMINE THE PARTITION P. (P CONSISTS OF THE MEMBERS OF THE PARTITION OF $\{1, \dots, n-1\} - Y$, TOGETHER WITH $X = Y \cup \{n\}$.)

IF $|Y| = k-1$ THEN

$$|\{1,\ldots,n-1\} - Y| = (n-1)-(k-1) = n-k.$$

IN THIS CASE THERE ARE $C(n-1,k-1)$ WAYS TO CHOOSE Y, AND B_{n-k} WAYS TO CHOOSE A PARTITION OF $\{1,\ldots,n-1\} - Y$. THUS THERE ARE

$$C(n-1,k-1)\cdot B_{n-k}$$

WAYS TO MAKE THESE CHOICES IN SUCCESSION. NOW k CAN BE ANY INTEGER IN THE RANGE $1 \le k \le n$, SO BY THE SUM RULE

$$B_n = \sum_{k=1}^{n} C(n-1,k-1) B_{n-k}$$

AS CLAIMED.

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