7.5 EQUIVALENCE RELATIONS

DEFN

A RELATION ON A SET A IS CALLED AN EQUIVALENCE RELATION IF IT IS REFLEXIVE, SYMMETRIC, AND TRANSITIVE.

EX. THE $=$ RELATION ON ANY SET, e.g. $\mathbb{Z}$.

EX. $\equiv \pmod{m}$ ON $\mathbb{Z}$.

EX. LET A BE THE SET OF ALL TERNARY STRINGS. DEFINE $\ell : A \to \mathbb{N}$ BY

$$\ell(x) = \text{LENGTH OF THE STRING } x.$$

DEFINE

$$R = \{ (x,y) \in A \times A \mid \ell(x) = \ell(y) \}.$$

THEN R IS AN EQUIVALENCE RELATION ON A.

DEFN!

LET R BE AN EQUIVALENCE RELATION ON A, AND LET $a \in A$. THE EQUIVALENCE CLASS OF a IS THE SET

$$[a]_R = \{ b \in A \mid aRb \}.$$

WE MAY WRITE $[a]$ IF R IS UNDERSTOOD.

IF $b \in [a]$ WE SAY b IS A <u>REPRESENTATIVE</u> OF THE EQUIVALENCE CLASS $[a]$.

<u>EX.</u> $=$ ON $\mathbb{Z}$: $[n] = \{n\}$.

<u>EX.</u> $\equiv \pmod{m}$ ON $\mathbb{Z}$:

LET $m=3$. THEN

$$[0] = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$$

$$[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$$

$$[2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$$

IN GENERAL FOR ANY m:

$$[n] = \{k \mid k \equiv n \pmod{m}\} = \{n + dm \mid d \in \mathbb{Z}\}$$

$$= \{\ldots, n-2m, n-m, n, n+m, n+2m, n+3m, \ldots\}$$

THERE ARE m DISTINCT CONGRUENCE CLASSES MODULO m.

<u>EX.</u> A, $\ell : A \to \mathbb{N}$, R AS ABOVE.

$[010211201] = \{$ All TERNARY STRINGS OF LENGTH $9\}$.

$[x] = \{$ All TERNARY STRINGS OF LENGTH $\ell(x)\}$.

Ex. $R \subseteq \mathbb{R} \times \mathbb{R}$

$$R = \{ (x,y) \mid x - y \in \mathbb{Z} \}$$

(EXERCISE: SHOW R IS AN EQUIVALENCE RELATION.)

$$[x] = \{ x + k \mid k \in \mathbb{Z} \}$$

$$\left[\tfrac{1}{2}\right] = \{ \ldots, -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \}$$

$$[\pi] = \{ \ldots, \pi - 1, \pi, \pi + 1, \pi + 2, \ldots \}$$

EXERCISE: DO THIS SAME EXAMPLE WITH $\mathbb{Z}$ REPLACED BY $\mathbb{Q}$.

THEOREM

LET R BE AN EQUIVALENCE RELATION ON A SET A. THE FOLLOWING STATEMENTS ARE EQUIVALENT.

(i) aRb
(ii) $[a] = [b]$
(iii) $[a] \cap [b] \neq \emptyset$

PROOF

(i) $\Rightarrow$ (ii)

SUPPOSE aRb. WE SHOW $[a] \subseteq [b]$ AND $[b] \subseteq [a]$, WHENCE $[a] = [b]$.

LET $c \in [a]$. THEN

aRc	(DEFN. OF EQ. CLASS)
cRa	(SYMMETRY)
cRb	(TRANSITIVITY)
bRc	(SYMMETRY)
$c \in [b]$	(DEFN. OF EQ. CLASS)

THUS $c \in [a] \rightarrow c \in [b]$, AND $\therefore$ $[a] \subseteq [b]$.
SIMILARLY $[b] \subseteq [a]$.

(ii) $\Rightarrow$ (iii)

SUPPOSE $[a]=[b]$. THEN $[a]\cap[b]=[a]$. THUS TO SHOW $[a]\cap[b]\neq\emptyset$ WE NEED ONLY SHOW $[a]\neq\emptyset$. BUT aRa BY REFLEXIVITY, SO $a\in[a]$ AND $\therefore$ $[a]\neq\emptyset$

(iii) $\Rightarrow$ (i)

SUPPOSE $[a]\cap[b]\neq\emptyset$. THEN THERE EXISTS $c\in A$ SUCH THAT BOTH $c\in[a]$ AND $c\in[b]$. THEN

$$\begin{array}{ll} aRc \wedge bRc & \text{(DEFN. OF EQ. CLASS)} \\ aRc \wedge cRb & \text{(SYMMETRY)} \\ aRb & \text{(TRANSITIVITY)} \end{array}$$

WE'VE SHOWN (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). THUS ALL THREE STATEMENTS ARE logically EQUIVALENT.

OBSERVE THAT THE EQUIVALENCE CLASSES OF AN EQUIVALENCE RELATION ON A HAVE THE following PROPERTIES:

(1) EACH EQUIVALENCE CLASS IS NON-EMPTY

$$aRa \rightarrow a\in[a] \rightarrow [a]\neq\emptyset.$$

(2) ANY TWO EQUIVALENCE CLASSES ARE EITHER DISJOINT OR EQUAL.

LAST THM: (iii) $\Rightarrow$ (ii)

(3) THE UNION OF THE EQUIVALENCE CLASSES IS THE WHOLE SET A.

$$a \in [a] \Rightarrow \{a\} \subseteq [a]$$

$$\therefore \; A = \bigcup_{a\in A} \{a\} \subseteq \bigcup_{a \in A} [a] \subseteq A$$

$$\therefore \; A = \bigcup_{a\in A} [a].$$

DEFN

LET S BE A SET AND $P = \{S_1, S_2, \ldots, S_n\}$ A COLLECTION OF SUBSETS OF S.

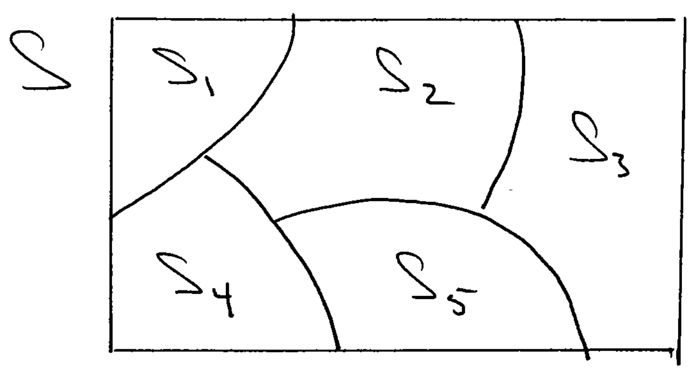
$$S_i \subseteq S \qquad (1 \leq i \leq n)$$

WE SAY THAT P IS A PARTITION OF S IF THE FOLLOWING PROPERTIES ARE SATISFIED:

(1) EACH MEMBER OF $\mathcal{P}$ IS NON-EMPTY:
$S_i \neq \phi \quad (1 \le i \le n)$.

(2) ANY TWO MEMBERS OF $\mathcal{P}$ ARE EITHER DISJOINT OR EQUAL: $S_i \cap S_j \neq \phi \Leftrightarrow i = j$.

(3) THE UNION OF THE MEMBERS OF $\mathcal{P}$ IS ALL OF S: $\bigcup_{i=1}^{n} S_i = S$.



THE SUM RULE CAN NOW BE STATED SUCCINCTLY AS

$$|S| = \sum_{i=1}^{n} |S_i|.$$

WE'VE SEEN THAT THE SET OF EQUIVALENCE CLASSES ASSOCIATED WITH AN EQUIVALENCE RELATION ON S FORM A PARTITION OF S.