10/28 ①

Special class of models called <u>linear</u> (time-invariant).

$$X_{k+1} = a X_k$$

for some constant a.

<u>Q1.</u> What are the fixed point(s) of this model? (Task. 3.4.2)

<u>Kennedy's notes:</u>

1. Plug in X_* for X_k, X_{k+1}

$$x_* = a x_*$$

2. Solve for X_*

$$a = \frac{x_*}{x_*} \qquad a = 1 \qquad X_* = ?$$

$\underline{X_{k+1} = X_k} \Rightarrow$ X_* is any real number

$$x_* = a x_* \iff \boxed{x_*(1-a) = 0}$$

OR $\underline{x_*(a-1) = 0}$

$$a = 1 \longrightarrow \underline{\forall\, x_* \in \mathbb{R}}$$

$$a \neq 1 \longrightarrow \underline{x_* = 0}$$

Assume $a \neq 1$

so $x_{k+1} = a x_k$ has

1 fixed point: $x_* = 0$.

Q. what is the stability of $x_* = 0$?

$\boxed{a = 1/2}$

orbit starting from $x_1 = 1$

$x_2 = a \cdot x_1 = \frac{1}{2} \cdot 1 = 1/2$

$x_3 = 1/4 = a \cdot x_2 = \frac{1}{2} \cdot \left(\frac{1}{2}\right)$

$x_4 = \frac{1}{2} \cdot x_3 = \frac{1}{8} \Rightarrow x_* = 0$ is S. & A.

$$\boxed{X_{k+1} = a X_k}$$

$$X_2 = a \cdot X_1$$

$$X_3 = a \cdot X_2 = a^2 \cdot X_1$$

$$X_4 = a \cdot X_3 = a^3 \cdot X_1$$

$$\vdots$$

$$\underline{X_n = a^{n-1} X_1}$$

Q. Does $\lim_{n\to\infty} X_n = X_* = 0$?

$$\boxed{X_1 \left(\lim_{n\to\infty} a^{n-1} \right) = \quad ?}$$

If $\underline{|a| < 1}$

$$\lim_{n\to\infty} a^n = 0$$

So $X_n \xrightarrow{n\to\infty} 0$

and $X_* = 0$ is <u>Stable & attractive</u>

<u>Examples:</u> $X_{k+1} = 0 \cdot X_k = 0$

$X_{k+1} = \frac{1}{2} X_k$

$X_{k+1} = 0.99995\, X_k$

If $|a| > 1$

$X_* = 0$ is <u>unstable</u>

Examples: $X_{k+1} = 3 X_k$

0.7931

An analytic test for stability of a fixed point

$x_{k+1} = f(x_k)$, x_*

If $\left| \frac{df}{dx}(x = x_*) \right| < 1$

$\Rightarrow$ stable & attractive

If $\left| \frac{df}{dx}(x = x_*) \right| > 1$

$\Rightarrow$ unstable

Settling time - helps determine <u>How</u> stable & attractive a f.p. is. (the degree of attractivity)

<u>Example</u> $x_{k+1} = \cos(x_k)$, $\underbrace{x_* = 0.739}_{\text{stable \& attractive}}$

what is the settling time to get the orbit from 0 to $x_* \pm 0.05$