Question	1	2	3	4	5	Total	Percent
Score							
Out of	15	15	15	15	15	75	100

Name Nic

AMS 10 Midterm Exam

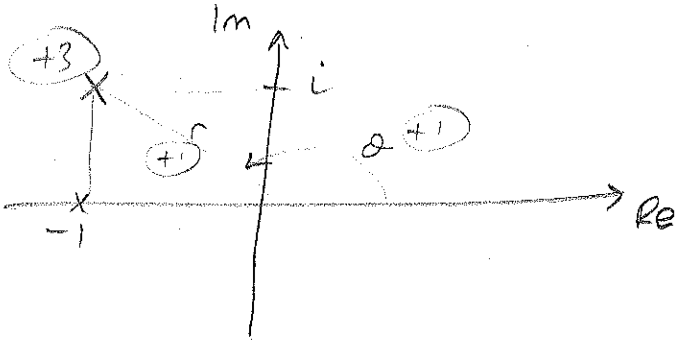
Section day (Mon Wed or Fri) ______________

Spring Quarter 2009

1. **Complex numbers** (15 points total)

(a) (5 points) Sketch the Argand diagram to show where the complex number $z_0 = -1 + i$ is located.
ANSWER:

Indicate r, θ of polar form on your diagram



(b) (5 points) Convert z_0 to polar form.
ANSWER:

$z_0 = -1 + i$

$r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$ +2

$\theta = \pi/2 + \pi/4$ (from sketch above) $= 3\pi/4$ +2

$\Rightarrow z_0 = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$ +1

(or $z_0 = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$ or $z_0 = \sqrt{2}\,e^{i\left(\frac{3\pi}{4}\right)}$)

(c) (5 points) Solve $z^4 = z_0$
ANSWER:

$z^4 = z_0 = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4} + 2k\pi\right)$

$\Rightarrow z = (\sqrt{2})^{1/4}\left(\mathrm{cis}\left(\frac{3\pi}{4} + 2k\pi\right)\right)^{1/4}$ +2

$= 2^{1/8}\,\mathrm{cis}\left(\frac{3\pi}{16} + \frac{2k\pi}{4}\right)$ +2

$\Rightarrow$ solns: $z_k = 2^{1/8}\,\mathrm{cis}\left(\frac{3\pi + 8k\pi}{16}\right)$ $k = 0, 1, 2, 3$ +1

$\left(\frac{3\pi}{16}, \frac{11\pi}{16}, \frac{19\pi}{16}, \frac{27\pi}{16}\right)$

2. **Linear systems** (15 points total)

This question concerns the following linear system:

$$\begin{aligned} x_1 + 4x_2 - 4x_3 + 4x_4 &= 5 \\ 2x_1 - x_2 + x_3 - x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \end{aligned}$$

(a) (2 points) Write the system as a matrix equation $\mathbf{Ax} = \mathbf{b}$ and construct the augmented matrix $\mathbf{M} = [\mathbf{A}|\mathbf{b}]$ for this system.

ANSWER:

$$\underbrace{\begin{bmatrix} 1 & 4 & -4 & 4 \\ 2 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}}_{b} \quad +1$$

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 4 & -4 & 4 & 5 \\ 2 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 2 \end{array}\right]}_{M} \quad +1$$

(b) (5 points) Reduce the augmented matrix to *reduced row echelon form*. — and circle the pivots

ANSWER:

$$\left[\begin{array}{cccc|c} 1 & 4 & -4 & 4 & 5 \\ 2 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 2 \end{array}\right] \quad +1$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -4 & 4 & 5 \\ 0 & -9 & 9 & -9 & -9 \\ 0 & -3 & 3 & -3 & -3 \end{array}\right] \begin{array}{l} 1, R2 - 2R1 \\ 2, R3 - R1 \end{array} \quad +1$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -4 & 4 & 5 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} 1, \text{Scale } R2: \div \text{ by } -9 \\ 2, \text{Scale } R3: \div \text{ by } -3 \end{array} \quad +1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} 2, R1 - 4R2 \\ 1, R3 - R2 \end{array} \quad +1$$

pivots. +1

(question continued on next page)

(c) (2 point 2) What is the rank of **A** and the rank of **M**?
ANSWER:

rank $A = 2$ rank $M = 2$

+1   +1

(d) (1 point) How many solutions are there to this system (0,1, or ∞)?
ANSWER:

∞ +1

(e) (1 point) Which variables are free parameters (if any)?
ANSWER:

x_3 and x_4 +1

(f) (4 points) Write out the solution (x_1, x_2, x_3, x_4) to the linear system.
ANSWER:

$x_4 = t$ (free parameter)   +1

$x_3 = s$ (free parameter)   +1

Row 2 $\Rightarrow$ $\emptyset x_1 + 1x_2 - 1x_3 + 1x_4 = 1$

$\Rightarrow x_2 - s + t = 1$

$\Rightarrow x_2 = 1 + s - t$ +1

Row 1 $\Rightarrow$ $1x_1 + \emptyset x_2 + \emptyset x_3 + \emptyset x_4 = 1$

$\Rightarrow x_1 = 1$ +1

$\Rightarrow$ solution:

$$(x_1, x_2, x_3, x_4) = (1,\ 1+s-t,\ s,\ t)$$

(or $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1+s-t \\ s \\ t \end{bmatrix}$)

3. **Applications** (15 points total) A company makes three different assortments of nuts - Regular, Superior and Super-Delicious. Each assortment is assembled from a certain ratio of peanuts, walnuts and cashew nuts as shown in the table below, along with the price paid per pound for each of the assortments.

	Peanuts	Walnuts	Cashews	Price
Regular	40%	60%		\$1.80
Superior	40%	40%	20%	\$4.00
Super-delicious	20%	50%	30%	\$4.40

How much does a pound of each of the different types of nuts cost?

(Hint: Gauss elim & back subst. is less work than Gauss-Jordan elim!)

ANSWER:

Let x_1 = price per pound of peanuts
x_2 = " walnuts
x_3 = " cashews.

Linear system:

$$\begin{bmatrix} 0.4 & 0.6 & 0.0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.80 \\ 4.00 \\ 4.40 \end{bmatrix}$$

+5 for setting up system

or rescaling

$$\begin{bmatrix} 4 & 6 & 0 \\ 4 & 4 & 2 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 40 \\ 44 \end{bmatrix}$$

Reduce:

$$\left[\begin{array}{ccc|c} 4 & 6 & 0 & 18 \\ 0 & -2 & 2 & 22 \\ 0 & 2 & 3 & 35 \end{array}\right] \begin{array}{l} \\ 1,\ R2 - R1 \\ 2,\ R3 - \tfrac{1}{2} R1 \end{array}$$

+5 for forward elimination

$$\left[\begin{array}{ccc|c} 4 & 6 & 0 & 18 \\ 0 & -2 & 2 & 22 \\ 0 & 0 & 5 & 57 \end{array}\right] 1,\ R3 + R2$$

+5 for back sub

Back subs: Row 3 $\Rightarrow$ $5x_3 = 57$ $\boxed{x_3 = 57/5}$

Row 2 $\Rightarrow$ $-2x_2 + 2x_3 = 22$
$\Rightarrow$ $x_2 = \dfrac{22 - 2(57/5)}{-2} = \boxed{2/5}$

Row 3 $\Rightarrow$ $4x_1 + 6x_2 + \cancel{0}x_3 = 18$
$x_1 = \dfrac{18 - 6(2/5)}{4} = \boxed{\dfrac{39}{10}}$

$x_1 = \$3.90$
$x_2 = \$0.40$
$x_3 = \$11.40$

4. **Inverses of matrices** (15 points total) Find the solution of the following linear system by inverting the coefficient matrix.

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 12 \\ x_1 + 4x_2 + 3x_3 &= -10 \\ x_1 + 3x_2 + 4x_3 &= 16 \end{aligned}$$

ANSWER: System can be written as

$$\underset{\underline{A}}{\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}} \underset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\underline{b}}{\begin{bmatrix} 12 \\ -10 \\ 16 \end{bmatrix}}$$

Find inverse of $\underline{A}$:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

+5 for system and matrix to reduce

Reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} 1,\ R2-R1 \\ 2,\ R3-R1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \ 1,\ R1-3R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \ 1,\ R1-3R3$$

A^{-1}

+5 for Gauss-Jordan reduction to find A^{-1}

Solve $\underline{x} = \underline{A}^{-1}\underline{b}$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ -10 \\ 16 \end{bmatrix} = \begin{bmatrix} \overset{84}{(7\times 12)} + \overset{30}{(-3\times -10)} + \overset{-48}{(-3\times 16)} \\ -12 \quad -10 \\ -12 \quad +16 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} 66 \\ -22 \\ 4 \end{bmatrix}$$

+5 for multiplication to get answer

5. **MATLAB** (15 points total)

Write down the essential Matlab commands you would need to perform the following operations:

(a) (3 points) Find the roots of the following complex equation

$$3z^3 - 2iz^2 = 1 + 2i$$

ANSWER: $3z^3 - 2iz^2 + \emptyset z + (-1-2i) = 0$ is eqn.

MATLAB > a = [3, -2i, Ø, -1-2i] +2

roots (a) +1

(b) (3 points) Plot the real 2D plane defined by $z = x - y$ in 3D space on a 10×10 grid between [-5, 5] [-5, 5]

ANSWER:

MATLAB > [x, y] = meshgrid(-5:1:5, -5:1:5) +1

z = x - y +1

surf (x, y, z) or PLOT3 (x, y, z) +1

(c) (3 points) Find the reduced row echelon form of

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

ANSWER: MATLAB > a = [1 2 3 4; 5 6 7 8] +1

rref (a) +2

(d) (3 points) Find the inverse of

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

ANSWER: MATLAB > b = [1 2; 3 4] +1

inv (b) +2

(e) (3 points) Figure out what the upper and lower triangular matrices are that correspond to the matrix

$$C = \begin{pmatrix} 1 & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{7} & \cancel{8} & \cancel{9} \end{pmatrix} \begin{matrix} \\ 3 \; 2 \; 1 \\ 2 \; 1 \; 3 \end{matrix}$$

ANSWER:

MATLAB > c = [1 2 3; 3 2 1; 2 1 3] +1

[L, U] = lu(c) +2