Proof We use **(P5)** as follows:

$$|z+w|^2 = (z+w)\overline{(z+w)} = z\bar{z} + z\overline{w} + \bar{z}w + w\overline{w} = |z|^2 + z\overline{w} + \overline{(z\overline{w})} + |w|^2$$
$$= |z|^2 + 2Re(z\overline{w}) + |w|^2$$
$$\le |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2.$$

Take positive square roots to complete the proof.

Mathematical induction extends inequality (8.25) to three or more complex numbers. For any positive integer k, we have

$$|z_1 + z_2 + \cdots + z_k| \le |z_1| + |z_2| + \cdots + |z_k|.$$

EXERCISES 8.1

Exercises 1–12. Express each complex number in standard form.

1. $7+5i+2(3-4i)$
2. $7+4i+i(6-5i)$
3. $(7+5i)(3-4i)$
4. $(3-4i)(5-6i)$
5. $(\sqrt{2}+3i)(\sqrt{2}-4i)$
6. $(1+i)^4$
7. $(2+3i)^2$
8. $(\sqrt{3}+i\sqrt{2})(\sqrt{3}-i\sqrt{2})$
9. $(4+i)^3$
10. $(2+i)^3(2-i)$
11. $(3+4i)^2(4+3i)-(2+3i)^2$
12. $5i(1+i)$

Exercises 13–30. Rationalize denominators and express in standard form.

13. $(1+\frac{1}{i})^2 + (1-\frac{1}{i})^2$
14. $\dfrac{7-9i}{1+i}$
15. $\dfrac{1+i}{1-i}$
16. $\dfrac{5}{i}$
17. $\dfrac{5+3i}{8-2i}$
18. $\dfrac{2-3i}{1+2i}$
19. $\dfrac{(1+i)^2}{1-i}$
20. $\dfrac{1}{(4+2i)(2-3i)}$
21. $\dfrac{1+i}{1-i} + \dfrac{1-i}{1+i}$
22. $\dfrac{5+2i}{5-2i} + \dfrac{5-2i}{5+2i}$
23. $\dfrac{(1+2i)(4-5i)}{2+3i}$
24. $\dfrac{(4+i)(5-2i)}{(4-i)(5+2i)}$
25. $\dfrac{1}{(1+i)^2} + \dfrac{1}{(1-i)^2}$
26. $\dfrac{1}{(1+i)^4} + \dfrac{1}{(1-i)^4}$
27. $\dfrac{(\sqrt{3}+i)(1+i\sqrt{3})}{(1+i)^2}$
28. $\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^3$
29. $\dfrac{1}{i^3(3-4i)}$
30. $\left(\dfrac{1+i}{1-i}\right)^3$

Exercises 31–36. Solve for real numbers a and b.

31. $(2-i)a+(1+3i)b+2=0$
32. $a(5+4i)+b(4+3i)+1=0$
33. $(4+3i)a+(3-4i)b=7-i$
34. $2ia+(5+3i)b=11-5i$
35. $(5+3i)a+(2+5i)b=10+11i$
36. $\dfrac{5+ai}{8-2i} = \dfrac{b+4i}{5+3i}$

Exercises 37–38. Solve each linear system for the real unknowns x, y, z, w.

37. (S) $\begin{cases} (1+i)x+(1+2i)y+(1+3i)z+(1+4i)w = 1+5i \\ (3-i)x+(4-2i)y+(1+i)z+4iw = 2-i \end{cases}$

38. (S) $\begin{cases} (1+2i)x+(2+i)y+(3+2i)z+(4+3i)w = 5+i \\ (3+4i)x+(2+3i)y+(1+2i)z+(2+i)w = 1-5i \end{cases}$

Exercises 39–44. Find the modulus of each complex number.

39. $4+3i$
40. $1+i$
41. $-i$
42. -3
43. $2-i$
44. $\dfrac{(1+i)}{(1-i)}$

Exercises 45–50. Find all the complex numbers $z=a+ib$ that satisfy the given equation.

45. $z^2=2+2i$
46. $z^2=i$
47. $z^2=3+4i$
48. $z=\bar{z}^2$
49. $iz+|z|=2+i$
50. $z^2-iz=0$

Exercises 51–56. Solve each equation for the complex number z.

51. $2z^2+(1-7i)z=12$
52. $z^2-6z+25=0$
53. $z^2+2-2i\sqrt{3}=0$
54. $z^2-(2+i)z+(-1+7i)=0$
55. $z^2-(1-i)z-2i=0$
56. $z^4-2z^2+4=0$
57. Let $z=a+ib$ and $w=c+id$. Prove that
$$\overline{z\pm w}=\bar{z}\pm\overline{w}, \qquad \overline{zw}=\bar{z}\,\overline{w}, \qquad \overline{\left(\frac{z}{w}\right)}=\frac{\bar{z}}{\overline{w}}.$$
58. Write the complex number $z=\dfrac{1-it}{t-i}$ in standard form $z=x+iy$ and prove that $x^2+y^2=1$. Interpret geometrically.
59. Expand $\frac{1}{2}(1-i)((1-ix)^5+i(1+ix)^5)$ in powers of the real variable x.

Exercises 60–68. Compute each modulus.

60. $|5i(1+i)|$
61. $|(2+3i)(3-4i)|$
62. $\left|\dfrac{1}{i}\right|$
63. $|(1+i)^2(\sqrt{3}-i)^5|$
64. $|(2-i)^6|$
65. $\left|\left(\dfrac{6-3i}{2+i\sqrt{6}}\right)^6\right|$
66. $\left|\dfrac{(2-3i)^2}{(1+2i)^3}\right|$
67. $\left|\dfrac{(2-i)(2+i)}{(1-i\sqrt{3})(1+i\sqrt{3})}\right|$
68. $\left|\dfrac{(1+2i)^{12}}{(1-2i)^9}\right|$
69. Solve the equation $z^2=-8+6i$ for z.
70. Let $z=x+iy$ and compute $r=\left|\dfrac{z-i}{z+i}\right|$. Show that when $0<y$.
71. Let $z=x+iy$. Simplify $\left(\dfrac{z}{\bar{z}}\right)^2-\left(\dfrac{\bar{z}}{z}\right)^2$.
72. Let $z=\cos\theta+i\sin\theta$. Show that $|2z+z^2|^2=5+4$
73. Let z be a nonzero complex number whose imag part is nonzero. Show that $\dfrac{i(z-|z|)}{z+|z|}$ is purely real. $z-\bar{z}=2i\,\mathrm{Im}\,z$.
74. Consider distinct complex numbers z_1 and z_2 suc $|z_1|=|z_2|$. Prove that $\dfrac{z_1+z_2}{z_1-z_2}$ is purely imaginary.
75. Prove that $|z_1+z_2|=|z_1|+|z_2|$ if and only if $z_1=$ where $k\ge 0$ is a real number.
76. Let z_1, z_2, z_3 be any complex numbers. Sub $z_1=z_3-z_2$ into the triangle inequality and show $||z_3|-|z_2||\le|z_3-z_2|$.
77. Use standard inequalities to prove that if z_1 and complex numbers such that $z_1+z_2=1$, then $|z_1|+|$ 1, with equality holding if and only if z_1 and z_2 are real and nonnegative.
78. Suppose z_1, z_2, z_3 are complex numbers such that $|$ $|z_3|$. Prove that
$$\left|\frac{z_1}{z_2+z_3}\right|\le\frac{|z_1|}{||z_2|-|z_3||}.$$

USING MATLAB

Consult online help for the commands **conj**, **abs**, **roots** *related commands.*

79. Use the command **roots** to check hand calculatic Exercises 45–56. *Hint:* The polynomial z^2-2- expressed as the row vector $[1, 0, -2-2i]$.
80. Use MATLAB to check hand calculations in this ex set.

8.2 Geometric Theory

The Complex Plane

We visualize complex numbers geometrically as points in the *complex p* Refer to Figure 8.2. Imagine the plane with a rectangular coordinate sy as the frame of reference and axes labeled Re and Im. This is calle

EXERCISES 8.2

1. Locate each complex number in the complex plane.

(i) 1 (ii) -1 (iii) $3i$
(iv) $-i$ (v) $1-i$ (vi) $2+2i$
(vii) $\frac{5}{i}$ (viii) $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ (ix) $\frac{1}{1+i}$

2. Let P represent the complex number $z = 1 - i$. Locate each of the following complex numbers in the complex plane.

(i) $2z$ (ii) $-3z$ (iii) $z+2$
(iv) $z-6$ (v) $3z+7$ (vi) $6-2z$
(vii) $z-2-i$ (viii) $z-4-3i$ (ix) z^2

3. For each given complex number z, locate the points z and $1/z$ in the complex plane.

(i) 2 (ii) i (iii) $-i$ (iv) $1+i$ (v) $3-4i$

4. Indicate the region in the complex plane corresponding to all points z that satisfy the given condition.

(i) $|z| = 11$ (ii) $|z+3| = 2$
(iii) $|z-3+4i| = 5$ (iv) $|z-7-12i| = 13$
(v) $|z+1+i| = 4$ (vi) $|2z+1| = 3$
(vii) $|i-2z| = 5$ (viii) $1 < |z+2-3i| < 2$
(ix) $\mathrm{Re}\, z > 4$

5. Suppose $|z| = 1$. In each case describe the set of points in the complex plane that correspond to the given expression.

(i) $3z$ (ii) $z+2$
(iii) $3z+7$ (iv) $\frac{1}{z}$

6. Find the maximum and minimum values of the given expression in z.

(i) $|z+3|$ such that $|z| \le 1$
(ii) $|z-4|$ such that $|z+3i| \le 1$

7. Let

$$w = f(z) = \frac{3z-1}{z-3}.$$

Show that the transformation $w = f(z)$ maps the unit circle onto itself. Draw two copies of the complex plane, labeling these the "z-plane" and "w-plane." Plot z and its image w when $z = 1, -1, i, -i$.

8. Show that the three numbers 1 and $(-1 \pm i\sqrt{3})/2$ form the vertices of an equilateral triangle inscribed in the circle $|z| = 1$. Draw a diagram. *Hint:* Use the vector properties of complex numbers.

9. Show that the complex number

$$z = \frac{1}{\cos\theta - 1 + i\sin\theta}$$

is defined whenever $\theta \neq 2\pi k$, where k is any integer. Compute the standard form $z = x + iy$ and prove that $x = -\frac{1}{2}$. Give a geometric description of all points z in the complex plane for which z is defined; that is, find the *locus* of all such points z.

10. Express the complex number

$$z = \frac{3}{2 + \cos\theta + i\sin\theta}$$

in standard form $z = x + iy$. Prove that $x^2 + y^2 = 4x - 3$ and show that all the numbers z (allowing θ to vary) lie on the circle in the complex plane, center $(2, 0)$ and radius $r = 1$.

11. If $w = z + \frac{1}{z}$ is real, show that either z is real or $|z| = 1$. *Hint:* consider $w - \overline{w}$.

12. Consider Example 2 and Figure 8.5.

(a) Find the points z such that $f(z) = z$. These points are *invariant* under the mapping f. Find points A and C such that A is mapped onto B and C onto D.

(b) Find the image under f of the points -1, i, $-i$, $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

USING MATLAB

Consult online help for the command **plot**, *and related commands.*

13. Write an M-file to plot the numbers in Exercise 1 in the complex plane.

14. Plot the equilateral triangle and unit circle in Exercise 8.

15. *Project.* Study the transformation $w = f(z) = \frac{1}{z-2}$. Choose a few points in the z-plane and plot their images in the w-plane. Plot the image in the w-plane of various regions of the z-plane such as the disk $|z| \le 1$ and the line $2\,\mathrm{Re}\, z + \mathrm{Im}\, z = -1$.

8.3 Polar Form

Refer to Figure 8.6. Consider a nonzero complex number $z = a+bi$ rep in the complex plane by the point P. We have $|z| = r > 0$. The vec also represents z and $r = |\overrightarrow{OP}|$. Let θ denote the angle formed by the positive real axis counterclockwise to coincide with the vector convention, the angle θ so formed is considered positive.

The length of the line segment AP is b, and so in $\triangle OAP$ v $a = r\cos\theta$ and $b = r\sin\theta$. Substituting into $z = a + ib$ (noting that i gives

$$z = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta).$$

We call (8.28) the *polar form* or the *modulus-argument form* of z.

For all θ, we have $\cos(-\theta) = \cos\theta$ (cosine is an even function o $\sin(-\theta) = -\sin\theta$ (sine is an odd function of θ), and so the polar forn complex conjugate $\overline{z} = a - bi$ is

$$\overline{z} = r(\cos\theta - i\sin\theta) = r\big(\cos(-\theta) + i\sin(-\theta)\big).$$

The angle θ shown in Figure 8.6 is not the only angle for which (8.28) Adding $2\pi, 4\pi, \ldots$ (positive revolutions) or $-2\pi, -4\pi, \ldots$ (negative tions) to θ gives the same value of z because

$$\cos(\theta \pm 2k\pi) = \cos\theta, \qquad \sin(\theta \pm 2k\pi) = \sin\theta$$

due to the periodicity of the trigonometric functions. Any angle chos the set $\theta \pm 2k\pi, k = 0, \pm1, \pm2, \ldots$ is called an *argument* or an *an* of z and denoted $\arg z$. We say that $\arg z$ is only determined up to an multiple of 2π radians.

The Principal Argument

For any nonzero complex number z there is a unique angle θ such tha

$$z = r(\cos\theta + i\sin\theta), \quad \text{where } -\pi < \theta \le \pi.$$

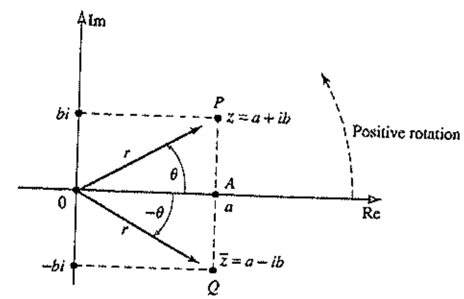

Figure 8.6 Polar form of a complex number.

Shorthand Notation

The notation $\cos\theta + i\sin\theta = \operatorname{cis}\theta$ (read "$\cos\theta$ plus $i\sin\theta$") is standard shorthand used in technical calculations. As with trigonometric functions, the meaning of $\operatorname{cis}^2\theta$, for example, is $(\operatorname{cis}\theta)^2$. Here are some of the previous results written in short form.

$$\operatorname{cis}\theta_1 \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1+\theta_2), \quad \frac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2} = \operatorname{cis}(\theta_1-\theta_2), \quad \overline{\operatorname{cis}\theta} = \operatorname{cis}(-\theta)$$

$$\frac{1}{\operatorname{cis}\theta} = \operatorname{cis}(-\theta), \qquad \operatorname{cis}^n\theta = \operatorname{cis}(n\theta) \quad \text{(de Moivre's theorem)}$$

EXAMPLE 5

Shorthand Notation

(a) $\dfrac{\operatorname{cis}^4(3\theta)(\operatorname{cis}2\theta)^{-3}}{\operatorname{cis}^5\theta\operatorname{cis}(4\theta)} = \dfrac{\operatorname{cis}(12\theta)\operatorname{cis}(-6\theta)}{\operatorname{cis}(5\theta)\operatorname{cis}(4\theta)} = \dfrac{\operatorname{cis}(6\theta)}{\operatorname{cis}(9\theta)} = \operatorname{cis}(-3\theta).$

(b)

$$\frac{(\sqrt{3}-i)^{14}}{(1-i)^3} = \frac{2^{14}\operatorname{cis}^{14}(-\pi/6)}{(\sqrt{2})^3\operatorname{cis}^3(-\pi/4)} = 2^{12}\sqrt{2}\,\frac{\operatorname{cis}(-7\pi/3)}{\operatorname{cis}(-3\pi/4)}$$

$$= 2^{12}\sqrt{2}\operatorname{cis}\left(-\frac{19\pi}{12}\right) = 2^{12}\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right). \quad ▪$$

Euler's Formula

Euler proved the fundamental identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{8.36}$$

where e is the exponential number. Hence any complex number z can be written in the compact form

$$z = re^{i\theta}. \tag{8.37}$$

Referring to Theorem 8.3 and the remark after the proof, we have

$$z = re^{i\theta} \quad \Rightarrow \quad z^n = r^n e^{in\theta}, \quad \text{for any integer } n.$$

Putting $z = -1$ in (8.37) gives the famous equation

$$e^{i\pi} + 1 = 0,$$

connecting five fundamental numbers of mathematics!

Infinite Series Expansions

Note that equation (8.36) follows from the infinite series expansions for e^z, $\cos\theta$, and $\sin\theta$.

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \quad \text{for real or complex } z.$$

Substituting $z = i\theta$ into e^z, we obtain (recall $0! = 1$)

$$e^{i\theta} = \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta,$$

using the series expansion for $\cos\theta$ and $\sin\theta$, namely

$$\cos\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} \quad \text{and} \quad \sin\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}.$$

Historical Notes

Abraham de Moivre (1667–1754). Born in France and educated in Belgium, de Moivre settled in England after fleeing from the persecution of the Huguenots (French Protestants) in France. His interest in mathematics started when, by accident, he saw a copy of *Principia Mathematica* by Sir Newton. De Moivre made important contributions to analytic geometry and the theory of probability through his publication *The Doctrine of Chance* (1718). However, his name is mostly remembered for the theorem that is stated in this section. Like Cardano, de Moivre is famous for predicting the actual day of his own death!

EXERCISES 8.3

1. Refer to Exercises 8.2, Problem 1(i)–(ix). Write the numbers in modulus-argument form.

2. Let α be any angle. Express each of the numbers in modulus-argument form.

   (i) $\cos\alpha - i\sin\alpha$ (ii) $\sin\alpha + i\cos\alpha$

   (iii) $\sin\alpha - i\cos\alpha$

3. If α is the principal argument of z, show that the principal argument of $\bar{z}$ is $-\alpha$.

4. Let $z = \cos\theta + i\sin\theta$. Express the following complex numbers in terms of θ.

   (i) $z + \dfrac{1}{z}$ (ii) $z - \dfrac{1}{z}$ (iii) $\dfrac{1}{\bar{z}}$

Exercises 5–22. Write in modulus-argument form.

5. $\left(\frac{1}{\sqrt{2}}(-1+i)\right)^4$
6. $(i-\sqrt{3})(1+i)^7$
7. $(1+i\sqrt{3})^6(1-i)^4$
8. $i^3(1-i)^{17}$
9. $(1+i)^2(\sqrt{3}-i)^5$
10. $(1+i)^5+(1-i)^5$
11. $\left(1+\dfrac{i}{\sqrt{3}}\right)^8 + \left(1-\dfrac{i}{\sqrt{3}}\right)^8$
12. $(\sqrt{3}-1)^{382}$
13. $\left(\dfrac{i-\sqrt{3}}{1-i}\right)^8$
14. $\left(\dfrac{\sqrt{3}+i}{1-i}\right)^{12}$
15. $\dfrac{(\sqrt{3}+i)^9}{(1+i\sqrt{3})^6}$
16. $\left(\dfrac{i}{1-i\sqrt{3}}\right)^6$
17. $\left(\dfrac{-1+i}{i+\sqrt{3}}\right)^9$
18. $\dfrac{(2+2i)^8}{(1-i\sqrt{3})^6}$
19. $\left(\dfrac{1+i\sqrt{3}}{\sqrt{3}-i}\right)^{18}$
20. $\left(\dfrac{i-\sqrt{3}}{1-i}\right)^5$
21. $\dfrac{(1-i)^8}{(i-\sqrt{3})^9}$
22. $\left(\dfrac{1-\sqrt{3}i}{1+i}\right)^7$

xercises 23–32. Write in polar form.

3. $\dfrac{1}{\cos 3\theta + i \sin 3\theta}$

24. $\dfrac{\cos 2\alpha + i \sin 2\alpha}{\cos\alpha + i \sin\alpha}$

5. $\dfrac{\cos\beta - i\sin\beta}{\cos\beta + i\sin\beta}$

26. $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta - i \sin 3\theta}$

7. $(\cos\theta - i\sin\theta)^3$

28. $\dfrac{(\cos\theta - i\sin\theta)^2}{(\cos\theta + i\sin\theta)^3}$

). $\dfrac{(\text{cis}\,5\theta)^3(\text{cis}\,\theta)^{-3}}{(\text{cis}\,2\theta)^5(\text{cis}\,3\theta)^2}$

30. $(\sin\frac{\pi}{5} + i\cos\frac{\pi}{5})^3$

l. $(\sin\theta - i\cos\theta)^3$

. $(\cos(\beta+\frac{\pi}{2}) + i\sin(\beta+\frac{\pi}{2}))^4$

. The function $w = 1/z$, defined for $z \neq 0$, is a fundamental transformation in the theory of complex numbers. Note that points outside the unit circle are mapped to points inside the unit circle, and vice versa. Points on the unit circle are mapped to points on the unit circle. Study the action of this transformation by locating images of points $z = 1, -1, i, -i, (-1-i)/2, 1+\sqrt{3}i$ in the complex plane.

USING MATLAB

Consult online help for the command **polar**, and related commands.

34. Write an M-file to plot a given complex number z (input) in standard and polar form. Test your program and use it to check hand calculations in Exercise 1 and Exercises 5–22 in this exercise set.

.4 Extraction of Roots, Polynomials

We will consider the solution of the fundamental equation

$$z^n = a_0, \quad \text{where } n = 1, 2, 3, \dots, \tag{8.38}$$

where a_0 is a given complex number and z is a complex variable. The method of solution uses the polar forms already introduced.

Let $a_0 = r_0(\text{cis}\,\alpha)$ be a fixed nonzero complex number, where $r_0 = |a_0|$ and $\alpha = \text{Arg}\,a_0$, and let $z = r(\text{cis}\,\theta)$. Using de Moivre's formula, equation (8.38) becomes

$$r^n(\text{cis}\,n\theta) = r_0(\text{cis}\,\alpha). \tag{8.39}$$

Equating the modulus and arguments on both sides of (8.39), we have

$$r^n = r_0 \quad\text{and}\quad n\theta = \alpha + 2k\pi, \quad \text{where } k = 0, \pm1, \pm2, \dots$$

because the difference $n\theta - \alpha$ can only be determined up to a multiple of 2π. Hence

$$r = \sqrt[n]{r_0}, \quad\text{and}\quad \theta = \frac{\alpha}{n} + \frac{2k\pi}{n}. \tag{8.40}$$

Notice that as k runs through the integers, (8.40) defines only n distinct complex numbers in the complex plane whose arguments are indexed by the integers 0 to $n-1$. We have

$$\theta_0 = \frac{\alpha}{n}, \quad \theta_1 = \frac{\alpha}{n} + \frac{2\pi}{n}, \quad \dots, \quad \theta_{n-1} = \frac{\alpha}{n} + \frac{2(n-1)\pi}{n}. \tag{8.41}$$

Observe that the value $k = n$ in (8.40) gives $\theta_0 + 2\pi$; that is, θ_0 plus a revolution, and $k = -1$ gives $\theta_{n-1} - 2\pi$; that is, θ_{n-1} minus a revolution, and so on. The n values for θ in (8.41) define the n-th *roots* of a_0 in (8.38), which are

$$z_k = \sqrt[n]{r_0}\left(\text{cis}\,\frac{(\alpha + 2k\pi)}{n}\right), \quad k = 0, 1, 2, \dots, n-1. \tag{8.42}$$

The nth roots of a_0 all lie on the circle C in the complex plane, with cente the origin and radius $\sqrt[n]{r_0}$. If the basic root $z_0 = \sqrt[n]{r_0}\,\text{cis}(\alpha/n)$ is located on the other roots are located on C by adding multiples of $2\pi/n$ to α/n, wh is the argument of z_0. The n-th roots are equally spaced around the circle (

When $a_0 = 0$, equation (8.38) has only one (multiple) solution $z = 0$.

EXAMPLE 1

Fifth Roots

Find all the values of $(1-i)^{1/5}$. We have $1 - i = \sqrt{2}\,\text{cis}(-\pi/4)$. Hence,

$$z_k = (1-i)^{1/5} = 2^{1/10}\,\text{cis}\left(-\frac{\pi}{4} + \frac{2k\pi}{5}\right) = 2^{1/10}\,\text{cis}\left(\frac{(8k-1)\pi}{20}\right),$$

where $k = 0, 1, 2, 3, 4$. We therefore have five values:

$$z_0 = \sqrt[10]{2}\,\text{cis}\left(-\frac{\pi}{20}\right), \quad z_1 = \sqrt[10]{2}\,\text{cis}\left(\frac{7\pi}{20}\right), \quad z_2 = \sqrt[10]{2}\,\text{cis}\left(\frac{3\pi}{4}\right),$$

$$z_3 = \sqrt[10]{2}\,\text{cis}\left(\frac{23\pi}{20}\right), \quad z_4 = \sqrt[10]{2}\,\text{cis}\left(\frac{31\pi}{20}\right).$$

Writing z_3 and z_4 with principal arguments, we get

$$z_3 = \sqrt[10]{2}\,\text{cis}\left(-\frac{17\pi}{20}\right), \qquad z_4 = \sqrt[10]{2}\,\text{cis}\left(-\frac{9\pi}{20}\right).$$

The numbers z_0, z_1, z_2, z_3, z_4 are equally spaced around a circle, center t origin and radius $\sqrt[10]{2}$. The angle between vectors representing consecuti numbers is $2\pi/5$.

An important special case arises when $a_0 = 1 = \text{cis}\,0$. Here $\alpha = \text{Arg}\,a_0$ 0 and the n roots are called the n-th *roots of unity*, given as follows:

$$\sqrt[n]{1} = \text{cis}\,\frac{2\pi k}{n}, \quad k = 0, 1, 2, \dots, n-1. \tag{8.4}$$

The root with $k = 1$ in (8.43), traditionally denoted by ω (Greek letter *omega* is

$$\omega = \text{cis}\,\frac{2\pi}{n}.$$

Using de Moivre's theorem, the roots in equation (8.43) are

$$\omega^0\ (=1), \quad \omega, \quad \omega^2, \quad \dots, \quad \omega^{n-1}.$$

Because ω satisfies the equation $\omega^n = 1$, the polynomial $\omega^n - 1$ has a fact $\omega - 1$ and dividing $\omega - 1$ into $\omega^n - 1$ gives a factorization

$$(\omega - 1)(\omega^{n-1} + \omega^{n-2} + \cdots \omega + 1) = 0.$$

But $\omega \neq 1$ and so

$$\omega^{n-1} + \omega^{n-2} + \cdots \omega + 1 = 0.$$

When plotted in the complex plane, the nth roots of unity form the vertice of a regular polygon of n sides, inscribed in the unit circle $|z| = 1$, with on vertex at $z = 1$. Figure 8.9 shows the polygons when $n = 3$ and $n = 6$.

Suppose the coefficients of $p(z)$ are real. We say that the roots of $p(z)$ appear in *complex conjugate pairs*. If z_0 is a (complex) root of $p(z)$, then $z - z_0$ and $z - \overline{z_0}$ are both factors of $p(z)$. Multiplying these factors, we obtain

$$(z - z_0)(z - \overline{z_0}) = z^2 - (z_0 + \overline{z_0})z + z_0\overline{z_0}. \qquad (8.46)$$

But $z_0 + \overline{z_0} = 2\,\mathrm{Re}(z_0)$ and $z_0\overline{z_0} = |z_0|^2$ are both real quantities. Hence, the quadratic in (8.46) has real coefficients and is a factor of $p(z)$. To summarize, we will make a formal statement.

Theorem 8.7 **Factoring Polynomials**

A polynomial of degree $n \geq 1$ with real coefficients can be written as a product of linear and quadratic factors with real coefficients.

XAMPLE 3 **Expressing a Polynomial as a Product of Factors**

The roots of the equation $z^5 - 1 = 0$ are $\operatorname{cis} \frac{2k\pi}{5}$, where $k = 0, 1, 2, 3, 4$.

These roots can be written in conjugate pairs as follows:

$$\operatorname{cis}\frac{2\pi}{5} \text{ and } \operatorname{cis}-\left(\frac{2\pi}{5}\right)\left(=\operatorname{cis}\frac{8\pi}{5}\right), \quad \operatorname{cis}\frac{4\pi}{5} \text{ and } \operatorname{cis}-\left(\frac{4\pi}{5}\right)\left(=\operatorname{cis}\frac{6\pi}{5}\right)$$

and one real root $\operatorname{cis} 0 = 1$.

Hence we have

$$z^5 - 1 = (z-1)\left(z - \operatorname{cis}\frac{2\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z - \operatorname{cis}\frac{4\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right).$$

■

AMPLE 4 **Finding Roots of a Polynomial**

Consider the polynomial

$$p(z) = z^4 - 6z^3 + 15z^2 - 18z + 10.$$

It so happens that the number $z = 2+i$ is a root of $p(z) = 0$. Then $\overline{z} = 2-i$ is also a root because the polynomial has real coefficients. Hence $(z-2-i)(z-2+i) = z^2 - 4z + 5$ is a factor and by long division,

$$z^4 - 6z^3 + 15z^2 - 18z + 10 = (z^2 - 4z + 5)(z^2 - 2z + 2).$$

The roots of $z^2 - 2z + 2$ are $1+i$ and $1-i$. Hence $p(z)$ has four roots: $2+i$, $2-i$, $1+i$, $1-i$.

■

CISES 8.4

-6. *Compute square roots of each number.*

2. $\cos 3\theta - i \sin 3\theta$
4. $-i$
6. 1

-12. *Compute cube roots of each number.*

8. 1
9. i
11. $1+i$
12. -1

Exercises 13–20. Find the required roots in polar form. Plot the roots in the complex plane.

13. Cube roots of $-2+2i$
14. Fourth roots of -4
15. Sixth roots of $-i$
16. Fourth roots of $-1+i\sqrt{3}$
17. Fourth roots of $-1+i$
18. Sixth roots of $1-i$
19. Cube roots of $\operatorname{cis}\frac{5\pi}{4}$
20. Fifth roots of 32

Exercises 21–30. Solve each equation for z.

21. $z^2 - (3-2i)z + (5-5i) = 0$
22. $z^4 - 30z^2 + 289 = 0$
23. $1 + iz^5 = i$
24. $z^7 - z^6 + iz - i = 0$
25. $z^4 + 8 + 8\sqrt{3}i = 0$
26. $z^9 - z = 0$
27. $z = iz^7$
28. $z^3 + 8 = 0$
29. $z^6 + i = 0$
30. $z^9 - iz^8 - z + i = 0$

Exercises 31–34. Express each polynomial as a product of linear or quadratic factors with real coefficients.

31. $z^3 + 8$
32. $z^4 + 16$
33. $z^6 - 64$
34. $z^5 + 1$
35. If $z + i$ is a root of $z^4 - 6z^3 - 15z^2 - 6z - 16$, find all the roots.
36. If $2+3i$ is a root of $z^4 - 6z^3 + 23z^2 - 34z + 26$, find all the roots.
37. If $z - \sqrt{2} - i\sqrt{3}$ is a factor of $z^4 - 2\sqrt{2}z^3 + 6z^2 - 2\sqrt{2}z + 5$, find all the factors.
38. Solve the equation $z^6 + 2z^3 + 4 = 0$. *Hint:* Substitute $Z = z^3$.
39. Let p and q be real numbers. Prove that, if $z = x + iy$ is a complex root of $z^3 + pz + q$, then x is a real root of $8z^3 + 2pz - q$. *Hint:* The root z is complex, meaning that $y \neq 0$.
40. The cubic equation $z^3 - 15z - 4 = 0$ appears in the introduction to this chapter and its roots are plotted in Figure 8.1. Factor the cubic and find its roots. Prove that the roots just found are exactly the roots given by the Cardaro–Tartaglia formula shown in (8.4), namely
$$z = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$$

USING MATLAB

41. Use the Symbolic Math Toolbox to solve Problems 21–30.
42. *Plotting Roots.* Let ω be a cube root of unity. Suppose $\omega \neq 1$. Prove that $1 + \omega + \omega^2 = 0$ and find $(1 + 2\omega + 3\omega^2)(1 + 3\omega + 2\omega^2)$. Verify your computation by plotting the complex numbers $1, 2\omega, 3\omega^2, (1+2\omega+3\omega^2), (1+3\omega+2\omega^2)$.

8.5 Linear Algebra: The Complex Case

In this section we return to linear algebra and review the main concepts in the case when scalars are complex numbers—we call this the *complex case*. The preceding chapters concentrated almost exclusively on the *real case* when scalars, coefficients, matrix entries, vector components, and so on are real numbers. The use of complex numbers as scalars was mentioned briefly in connection with eigenvalues.

The complex number system $\mathbb{C}$ is an extension of the real number system $\mathbb{R}$. Many concepts and definitions in linear algebra extend immediately from the real to the complex case without change, but there are striking exceptions and some adjustments are required. Certain concepts defined for real scalars need to be redefined in order to make sense in the complex case. The new definition, however, always agrees with the old one when all scalars are purely real.

The set of $m \times n$ matrices with complex entries is denoted by $\mathbb{C}^{m\times n}$. The space of n-vectors with complex components is denoted by $\mathbb{C}^n$. Note that $\mathbb{C}^n$ is identical to $\mathbb{C}^{n\times 1}$.

Matrices

An $m \times n$ matrix $\mathbf{A}$ is called *complex* if its entries are complex numbers. Of course, any real matrix (with purely real entries) is also complex. For example, the matrix $\mathbf{A}$, shown next, belongs to $\mathbb{C}^{2\times 3}$, and $\mathbf{v}$ is a vector in $\mathbb{C}^2$.

$$\mathbf{A} = \begin{bmatrix} i & 2 & 1+i \\ 1-i & 3i & 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1+2i \\ 3 \end{bmatrix}$$